Polarization Charge Distributions

Consider a hunk of **dielectric** material with volume V.

Say this dielectric material is immersed in an **electric field** $\mathbf{E}(\bar{\mathbf{r}})$, therefore creating atomic **dipoles** with density $\mathbf{P}(\bar{\mathbf{r}})$.

Q: What **electric potential field** $V(\bar{\mathbf{r}})$ is created by these diploes?

A: We know that:

$$V(\bar{\mathbf{r}}) = \iiint_V \frac{\mathbf{P}(\bar{\mathbf{r}}') \cdot (\bar{\mathbf{r}}-\bar{\mathbf{r}}')}{4\pi\varepsilon_0 |\bar{\mathbf{r}}-\bar{\mathbf{r}}'|^3} dv'$$

But, it can be shown that (p. 135):

$$\begin{aligned} V(\bar{\mathbf{r}}) &= \iiint_V \frac{\mathbf{P}(\bar{\mathbf{r}}') \cdot (\bar{\mathbf{r}}-\bar{\mathbf{r}}')}{4\pi\varepsilon_0 |\bar{\mathbf{r}}-\bar{\mathbf{r}}'|^3} dv' \\ &= \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{-\nabla \cdot \mathbf{P}(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|} dv' + \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{\mathbf{P}(\bar{\mathbf{r}}') \cdot \hat{a}_n(\bar{\mathbf{r}})}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|} ds' \end{aligned}$$

where S is the **closed** surface that surrounds volume V, and $\hat{a}_n(\bar{\mathbf{r}})$ is the unit vector **normal** to surface S (pointing **outward**).

This complicated result is only important when we compare it to the electric potential created by **volume** charge density $\rho_v(\bar{r})$ and **surface** charge density $\rho_s(\bar{r})$:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho_s(\bar{r}')}{|\bar{r}-\bar{r}'|}\, ds'$$

If both volume and surface charge are present, the **total** electric potential field is:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv' + \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho_s(\bar{r}')}{|\bar{r}-\bar{r}'|}\, ds'$$

Compare this expression to the previous integral involving the **polarization vector** $\mathbf{P}(\bar{r})$. It is evident that the two expressions are equal if the following relations are true:

$$\rho_{vp}(\bar{r}) = -\nabla \cdot \mathbf{P}(\bar{r})$$

and

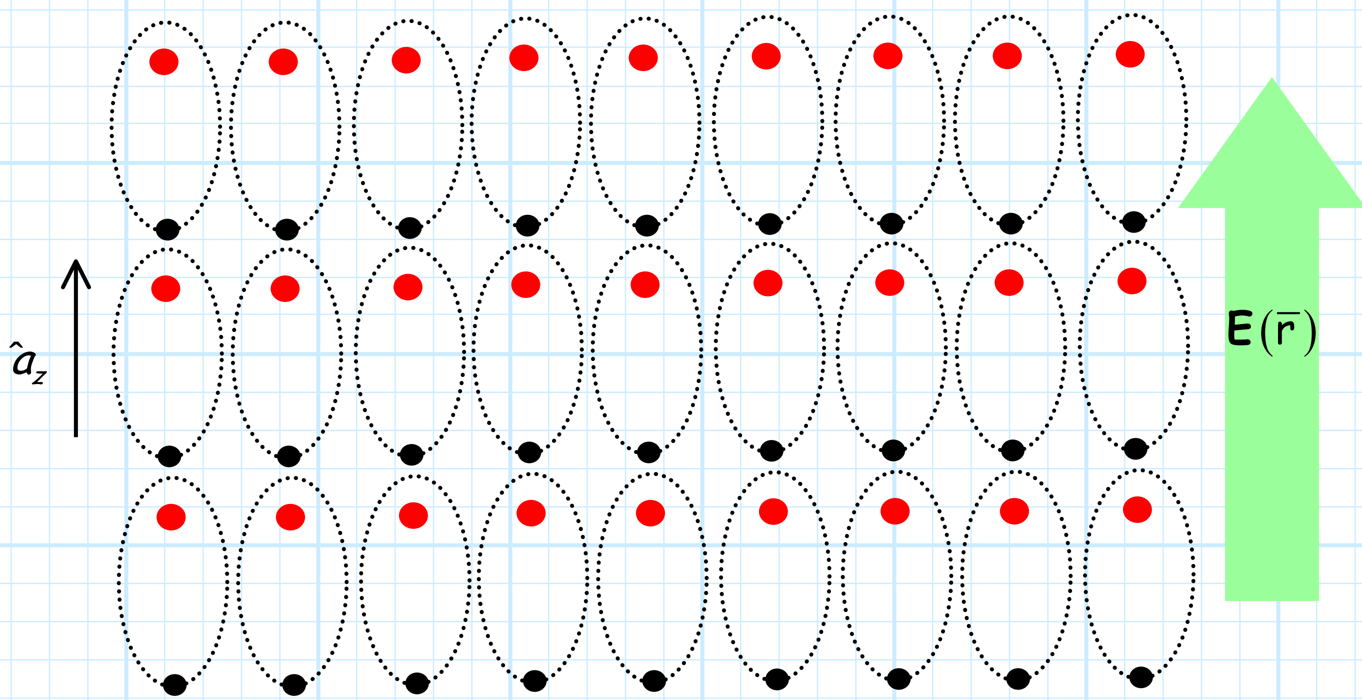
$$\rho_{sp}(\bar{r}) = \mathbf{P}(\bar{r}) \cdot \hat{a}_n$$

The subscript p (e.g., ρ_{vp}, ρ_{sp}) indicates that these functions represent **equivalent charge densities** due to the due to the **dipoles** created in the dielectric.

In other words, the electric potential field $V(\bar{r})$ (and thus electric field $\mathbf{E}(\bar{r})$) created by the dipoles in the dielectric (i.e., $\mathbf{P}(\bar{r})$) is **indistinguishable** from the electric potential field created by the equivalent charge densities $\rho_{vp}(\bar{r})$ and $\rho_{sp}(\bar{r})$!

For example, consider a dielectric material immersed in an electric field, such that its polarization vector $\mathbf{P}(\bar{r})$ is:

$$\mathbf{P}(\bar{r}) = 3\,\hat{a}_z \quad \left[\frac{C}{m^2}\right]$$



Note since the polarization vector is a **constant**, the equivalent volume charge density is **zero**:

$$\begin{aligned}\rho_{vp}(\bar{r}) &= -\nabla \cdot \mathbf{P}(\bar{r}) \\ &= -\nabla \cdot 3\,\hat{a}_z \\ &= 0\end{aligned}$$

On the **top** surface of the dielectric ($\hat{a}_n = \hat{a}_z$), the equivalent **surface** charge is:

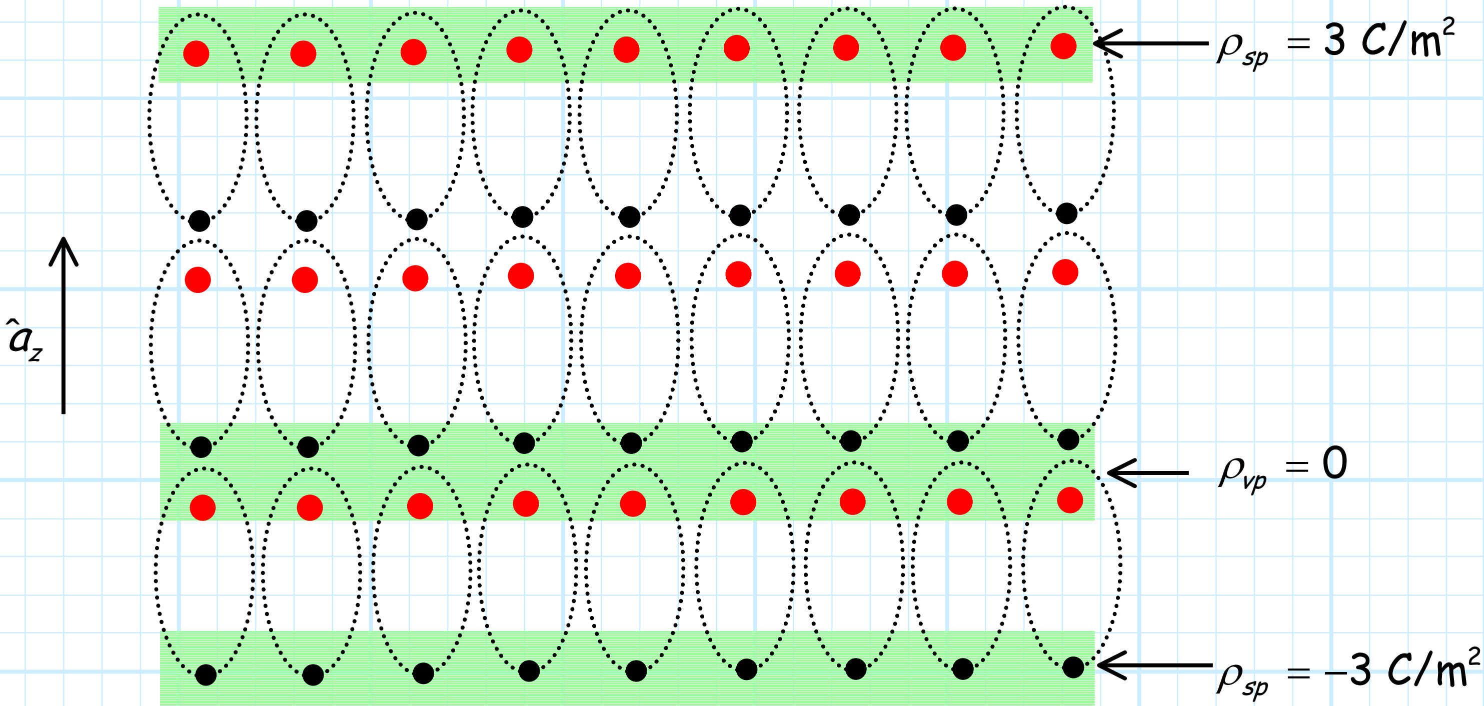
$$\begin{aligned}\rho_{sp}(\bar{r}) &= \mathbf{P}(\bar{r}) \cdot \hat{a}_n \\ &= 3\,\hat{a}_z \cdot \hat{a}_z \\ &= 3 \qquad \left[{C}/{m^2}\right]\end{aligned}$$

On the **bottom** of the dielectric ($\hat{a}_n = -\hat{a}_z$), the equivalent **surface** charge is:

$$\begin{aligned}\rho_{sp}(\bar{r}) &= \mathbf{P}(\bar{r}) \cdot \hat{a}_n \\ &= -3\,\hat{a}_z \cdot \hat{a}_z \\ &= -3 \qquad \left[{C}/{m^2}\right]\end{aligned}$$

On the **sides** of the dielectric material, the **surface** charge is **zero**, since $\hat{a}_z \cdot \hat{a}_n = 0$.

This result actually makes **physical** sense! Note at the **top** of dielectric, there is a layer of **positive** charge, and at the **bottom**, there is a layer of **negative** charge.



In the **middle** of the dielectric, there are **positive** charge layers on top of **negative** charge layers. The two add together and **cancel** each other, so that equivalent **volume** charge density is **zero**.

Finally, recall that there is no perfect dielectric, all materials will have some non-zero conductivity $\sigma(\bar{r})$.

As a result, we find that the total charge density within some material is the sum of the polarization charge density and the free charge (i.e., conducting charge) density:

$$\rho_{vT}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

Where:

$\rho_{vT}(\bar{r}) \doteq$ total charge density

$\rho_v(\bar{r}) \doteq$ free charge density

$\rho_{vp}(\bar{r}) \doteq$ polarization charge density

This is likewise (as well as **more frequently!**) true for **surface** charge density:

$$\rho_{sT}(\bar{r}) = \rho_s(\bar{r}) + \rho_{sp}(\bar{r})$$