Polarization Charge Distributions

Consider a hunk of dielectric material with volume V.

Say this dielectric material is immersed in an electric field \( E(\mathbf{r}) \), therefore creating atomic dipoles with density \( P(\mathbf{r}) \).

**Q:** What electric potential field \( V(\mathbf{r}) \) is created by these dipoles?

**A:** We know that:

\[
V(\mathbf{r}) = \iiint_{V} \frac{P(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3} \, dV'
\]

But, it can be shown that (p. 135):

\[
V(\mathbf{r}) = \iiint_{V} \frac{P(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3} \, dV' \\
= \frac{1}{4\pi\varepsilon_0} \iiint_{V} \frac{-\nabla \cdot P(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' + \frac{1}{4\pi\varepsilon_0} \iint_{S} P(\mathbf{r}') \cdot \hat{a}_n(\mathbf{r}) \, dS'
\]

where \( S \) is the closed surface that surrounds volume \( V \), and \( \hat{a}_n(\mathbf{r}) \) is the unit vector normal to surface \( S \) (pointing outward).
This complicated result is only important when we compare it to the electric potential created by volume charge density \( \rho_v(\vec{r}) \) and surface charge density \( \rho_s(\vec{r}) \):

\[
V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho_v(\vec{r}')}{|\vec{r}-\vec{r}'|} \, dv' 
\]

\[
V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{S} \frac{\rho_s(\vec{r}')}{|\vec{r}-\vec{r}'|} \, ds'
\]

If both volume and surface charge are present, the total electric potential field is:

\[
V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho_v(\vec{r}')}{|\vec{r}-\vec{r}'|} \, dv' + \frac{1}{4\pi\varepsilon_0} \int_{S} \frac{\rho_s(\vec{r}')}{|\vec{r}-\vec{r}'|} \, ds'
\]

Compare this expression to the previous integral involving the polarization vector \( P(\vec{r}) \). It is evident that the two expressions are equal if the following relations are true:

\[
\rho_{vp}(\vec{r}) = -\nabla \cdot P(\vec{r})
\]

and

\[
\rho_{sp}(\vec{r}) = P(\vec{r}) \cdot \hat{a}_n
\]
The subscript \( p \) (e.g., \( \rho_{vp}, \rho_{sp} \)) indicates that these functions represent **equivalent charge densities** due to the dipoles created in the dielectric.

In other words, the electric potential field \( V(\vec{r}) \) (and thus electric field \( E(\vec{r}) \)) created by the dipoles in the dielectric (i.e., \( P(\vec{r}) \)) is **indistinguishable** from the electric potential field created by the equivalent charge densities \( \rho_{vp}(\vec{r}) \) and \( \rho_{sp}(\vec{r}) \)!

For example, consider a dielectric material immersed in an electric field, such that its polarization vector \( P(\vec{r}) \) is:

\[
P(\vec{r}) = 3 \hat{a}_z \left[ \frac{C}{m^2} \right]
\]

![Diagram of dipoles and electric field](image)
Note since the polarization vector is a constant, the equivalent volume charge density is zero:

$$\rho_{vp}(\vec{r}) = -\nabla \cdot \mathbf{P}(\vec{r})$$
$$= -\nabla \cdot 3 \hat{a}_z$$
$$= 0$$

On the top surface of the dielectric ($\hat{a}_n = \hat{a}_z$), the equivalent surface charge is:

$$\rho_{sp}(\vec{r}) = \mathbf{P}(\vec{r}) \cdot \hat{a}_n$$
$$= 3 \hat{a}_z \cdot \hat{a}_z$$
$$= 3 \left[ \frac{C}{m^2} \right]$$

On the bottom of the dielectric ($\hat{a}_n = -\hat{a}_z$), the equivalent surface charge is:

$$\rho_{sp}(\vec{r}) = \mathbf{P}(\vec{r}) \cdot \hat{a}_n$$
$$= -3 \hat{a}_z \cdot \hat{a}_z$$
$$= -3 \left[ \frac{C}{m^2} \right]$$

On the sides of the dielectric material, the surface charge is zero, since $\hat{a}_z \cdot \hat{a}_n = 0$. 
This result actually makes physical sense! Note at the top of dielectric, there is a layer of positive charge, and at the bottom, there is a layer of negative charge.

In the middle of the dielectric, there are positive charge layers on top of negative charge layers. The two add together and cancel each other, so that equivalent volume charge density is zero.

Finally, recall that there is no perfect dielectric, all materials will have some non-zero conductivity $\sigma(\vec{r})$.

As a result, we find that the total charge density within some material is the sum of the polarization charge density and the free charge (i.e., conducting charge) density:
\[ \rho_{vT}(\vec{r}) = \rho_v(\vec{r}) + \rho_{vp}(\vec{r}) \]

Where:

\[ \rho_{vT}(\vec{r}) \doteq \text{total charge density} \]
\[ \rho_v(\vec{r}) \doteq \text{free charge density} \]
\[ \rho_{vp}(\vec{r}) \doteq \text{polarization charge density} \]

This is likewise (as well as more frequently!) true for surface charge density:

\[ \rho_{sT}(\vec{r}) = \rho_s(\vec{r}) + \rho_{sp}(\vec{r}) \]