5-4 Electrostatic Boundary Value Problems

Reading Assignment: *pp. 149-157*

Q:

A:

We must solve differential equations, **and** apply boundary conditions to find a **unique** solution.

In EE and CoE, we typically use a **voltage source** to apply boundary conditions on **electric potential function** $V(\bar{r})$.

This process is best demonstrated with a series of **examples:**

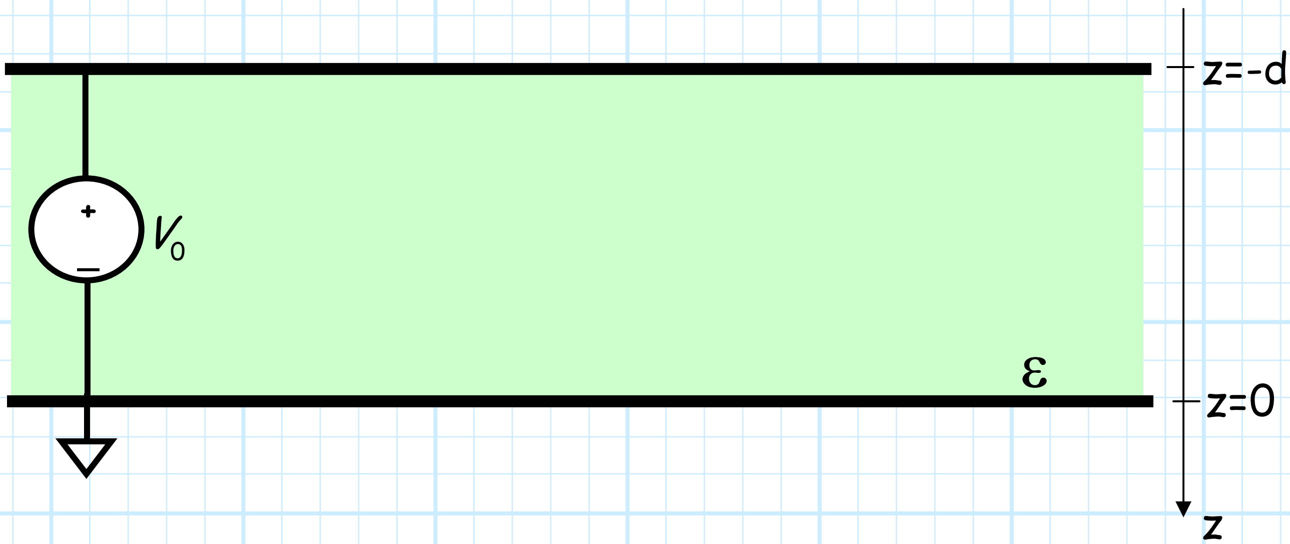
Example: Dielectric Filled Parallel Plates

Example: Charge Filled Parallel Plates

Example: The Electrostatic Fields of a Coaxial Line

Example: Dielectric Filled Parallel Plates

Consider two infinite, parallel **conducting** plates, spaced a distance d apart. The region between the plates is filled with a dielectric ε. Say a voltage V_0 is placed across these plates.



Q: *What electric potential field* $V(\bar{r})$*, electric field* $\mathbf{E}(\bar{r})$ *and charge density* $\rho_s(\bar{r})$ *is produced by this situation?*

A: We must solve a **boundary value problem** ! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Q: *Yikes! Where do we even start ?*

A: We might start with the electric potential field $V(\bar{r})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's equation**:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon}$$

b) It must also satisfy the **boundary conditions**:

$$V(z=-d) = V_0 \qquad\qquad V(z=0) = 0$$

Consider first the dielectric region ($-d < z < 0$). Since the region is a dielectric, there is **no** free charge, and:

$$\rho_v(\bar{r}) = 0$$

Therefore, Poisson's equation reduces to **Laplace's equation**:

$$\nabla^2 V(\bar{r}) = 0$$

This problem is greatly simplified, as it is evident that the solution $V(\bar{r})$ is independent of coordinates x and y. In other words, the electric potential field will be a function of coordinate z **only**:

$$V(\bar{r}) = V(z)$$

This make the problem **much** easier! Laplace's equation becomes:

$$\nabla^2 V(\bar{r}) = 0$$
$$\nabla^2 V(z) = 0$$
$$\frac{\partial^2 V(z)}{\partial z^2} = 0$$

Integrating **both** sides of Laplace's equation, we get:

$$\int \frac{\partial^2 V(z)}{\partial z^2}\, dz = \int 0\, dz$$
$$\frac{\partial V(z)}{\partial z} = C_1$$

And integrating **again** we find:

$$\int \frac{\partial V(z)}{\partial z}\, dz = \int C_1\, dz$$
$$V(z) = C_1 z + C_2$$

We find that the equation $V(z) = C_1 z + C_2$ **will** satisfy Laplace's equation (try it!). We must now apply the **boundary conditions** to determine the value of constants C_1 and C_2.

We know that the value of the electrostatic potential at every point on the top (z=-d) plate is $V(-d) = V_0$, while the electric potential on the bottom plate (z=0) is zero ($V(0) = 0$). Therefore:

$$V(z=-d) = -C_1 d + C_2 = V_0$$

$$V(z=0) = C_1(0) + C_2 = 0$$

Two equations and **two** unknowns (C_1 and C_2)!

Solving for C_1 and C_2 we get:

$$C_2 = 0 \quad \text{and} \quad C_1 = -\frac{V_0}{d}$$

and therefore, the **electric potential** field within the dielectric is found to be:

$$V(\bar{r}) = \frac{-V_0 z}{d} \qquad (-d \le z \le 0)$$

Before we proceed, let's do a **sanity check**!

In other words, let's evaluate our answer at $z = 0$ and $z = -d$, to make **sure** our result is correct:

$$V(z=-d) = \frac{-V_0(-d)}{d} \quad = \quad V_0 \quad \checkmark$$

and

$$V(z=0) = \frac{-V_0(0)}{d} \quad = \quad 0 \quad \checkmark$$

Now, we can find the **electric field** within the dielectric by taking the **gradient** of our result:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r}) = \frac{V_0}{d}\hat{a}_z \quad (-d \le z \le 0)$$

And thus we can easily determine the **electric flux density** by multiplying by the dielectric of the material:

$$\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r}) = \frac{\varepsilon V_0}{d}\hat{a}_z \quad (-d \le z \le 0)$$

Finally, we need to determine the **charge density** that actually created these fields!

Q: *Charge density !?! I thought that we already determined that the charge density $\rho_v(\bar{r})$ is equal to **zero**?*

A: We know that the free charge density **within the dielectric** is zero—but there must be charge **somewhere**, otherwise there would be no fields!

Recall that we found that **at** a conductor/dielectric **interface**, the **surface charge density** on the conductor is related to the **electric flux density** in the dielectric as:

$$D_n = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}}) = \rho_s(\bar{\mathbf{r}})$$

First, we find that the electric flux density on the **bottom** surface of the **top** conductor (i.e., at $z = -d$) is:

$$\mathbf{D}(\bar{\mathbf{r}})\Big|_{z=-d} = \frac{\varepsilon V_0}{d}\hat{a}_z\Bigg|_{z=-d} = \frac{\varepsilon V_0}{d}\hat{a}_z$$

For **every** point on **bottom** surface of the **top** conductor, we find that the unit vector **normal** to the conductor is:

$$\hat{a}_n = \hat{a}_z$$

Therefore, we find that the **surface charge density** on the bottom surface of the top conductor is:

$$\begin{aligned}\rho_{s+}(\bar{\mathbf{r}}) &= \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}})\Big|_{z=-d} \\ &= \hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} \\ &= \frac{\varepsilon V_0}{d} \qquad (z = -d)\end{aligned}$$

Likewise, we find the unit vector **normal** to the **top** surface of the **bottom** conductor is (do you see why):

$$\hat{a}_n = -\hat{a}_z$$

Therefore, evaluating the **electric flux density** on the top surface of the bottom conductor (i.e., $z = 0$), we find:

$$\begin{aligned}\rho_{s-}(\bar{r}) &= \hat{a}_n \cdot \mathbf{D}(\bar{r})\big|_{z=0} \\ &= -\hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} \\ &= \frac{-\varepsilon V_0}{d} \qquad (z = 0)\end{aligned}$$

We should **note** several things about these solutions:

1) $\nabla \times \mathbf{E}(\bar{r}) = 0$

2) $\nabla \cdot \mathbf{D}(\bar{r}) = 0$ and $\nabla^2 V(\bar{r}) = 0$

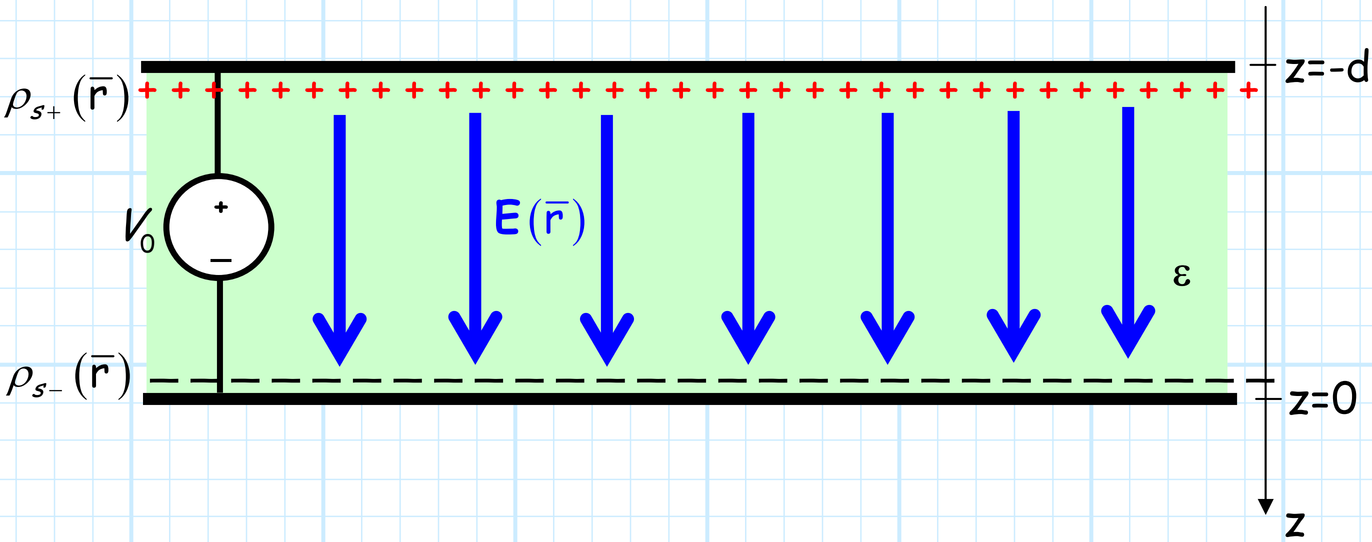
3) $\mathbf{D}(\bar{r})$ and $\mathbf{E}(\bar{r})$ are **normal** to the surface of the conductor (i.e., their **tangential** components are equal to **zero**).

4) The **electric field** is precisely the **same** as that given by using superposition and eq. 4.20 in section 4-5!

I.E.:

$$\mathbf{E}(\bar{r}) = \frac{\rho_{s+}}{2\varepsilon}\hat{a}_z - \frac{\rho_{s-}}{2\varepsilon}\hat{a}_z = \frac{V_0}{d}\hat{a}_z \quad (-d < z < 0)$$

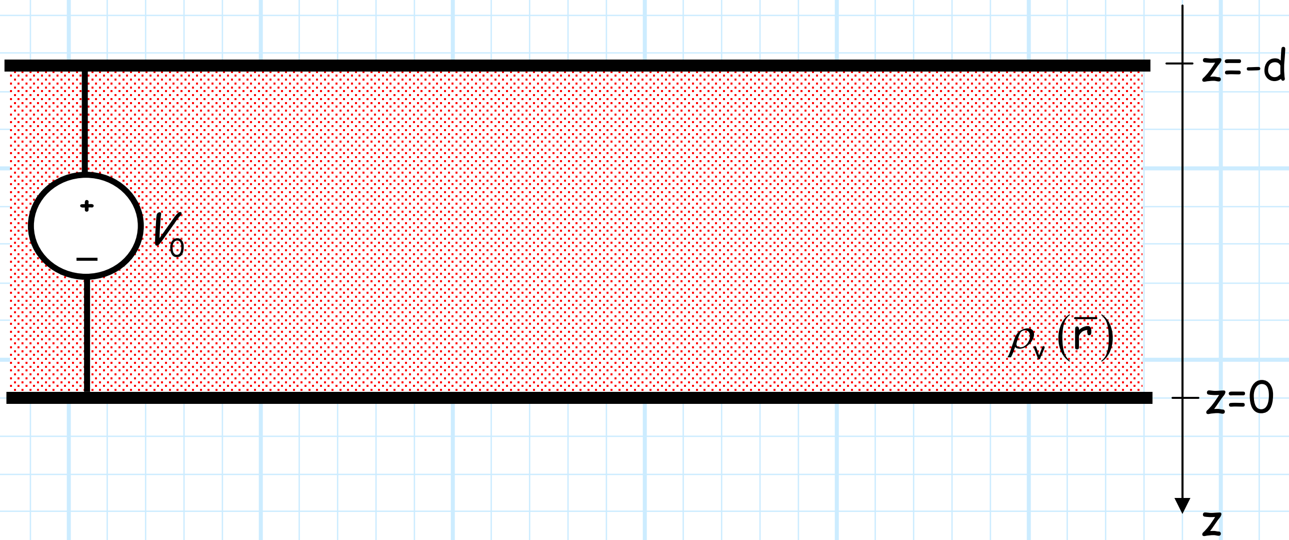
In other words, the **fields** $\mathbf{E}(\bar{r})$, $\mathbf{D}(\bar{r})$, and $V(\bar{r})$ are attributable to **charge densities** $\rho_{s+}(\bar{r})$ and $\rho_{s-}(\bar{r})$.



Example: Charge Filled Parallel Plates

Consider now a problem similar to the previous example (i.e., dielectric filled parallel plates), with the exception that the space between the infinite, conducting parallel plates is filled with **free charge**, with a density:

$$\rho_v(\bar{r}) = -z\,\varepsilon_0 \qquad (-d < z < 0)$$



Q: *How do we determine the fields within the parallel plates for **this** problem?*

A: Same as before! However, since the charge density between the plates is **not** equal to zero, we recognize that the electric potential field must satisfy **Poisson's equation**:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

For the specific charge density $\rho_v(\bar{\mathbf{r}}) = -z\ \varepsilon_0$:

$$\nabla^2 V(\bar{\mathbf{r}}) = \frac{-\rho_v(\bar{\mathbf{r}})}{\varepsilon_0} = z$$

Since **both** the charge density and the plate geometry are **independent** of coordinates x and y, we know the electric potential field will be a function of coordinate z **only** (i.e., $V(\bar{\mathbf{r}}) = V(z)$).

Therefore, Poisson's equation becomes:

$$\nabla^2 V(z) = \frac{\partial^2 V(z)}{\partial z^2} = z$$

We can solve this differential equation by first **integrating** both sides:

$$\int \frac{\partial^2 V(z)}{\partial z^2} dz = \int z\, dz$$

$$\frac{\partial V(z)}{\partial z} = \frac{z^2}{2} + C_1$$

And then integrating a **second time**:

$$\int \frac{\partial V(\bar{\mathbf{r}})}{\partial z} dz = \int \left(\frac{z^2}{2} + C_1 \right) dz$$

$$V(\bar{\mathbf{r}}) = \frac{z^3}{6} + C_1 z + C_2$$

Note that this expression for $V(\bar{r})$ **satisfies** Poisson's equation for this case. The question remains, however: what are the values of **constants** C_1 and C_2?

We find them in the same manner as before—**boundary conditions**!

Note the boundary conditions for **this** problem are:

$$V(z=-d)=V_0$$

$$V(z=0)=0$$

Therefore, we can construct **two** equations with **two** unknowns:

$$V(z=-d)=V_0=\frac{(-d)^3}{6}+C_1(-d)+C_2$$

$$V(z=0)=0=\frac{(0)^3}{6}+C_1(0)+C_2$$

It is evident that $C_2 = 0$, therefore constant C_1 is:

$$C_1=-\left(\frac{V_0}{d}+\frac{d^2}{6}\right)$$

The **electric potential field** between the two plates is therefore:

$$V(\bar{r}) = \frac{z^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right) z \qquad (-d < z < 0)$$

Performing our **sanity check**, we find:

$$\begin{aligned} V(z = \text{-d}) &= \frac{(-\text{d})^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right)(\text{-d}) \\ &= \frac{-\text{d}^3}{6} + V_0 + \frac{d^3}{6} \\ &= V_0 \quad \checkmark \end{aligned}$$

and

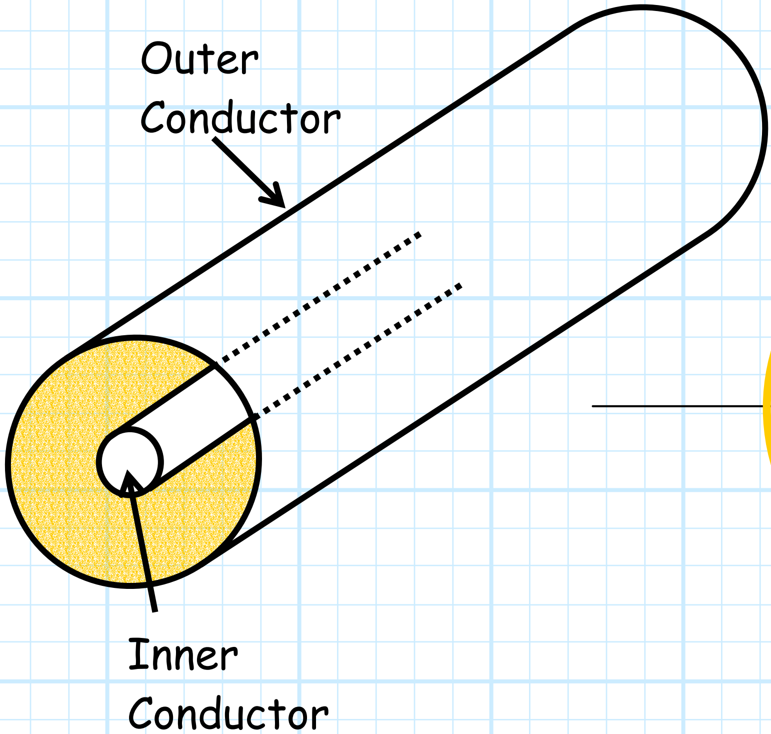
$$\begin{aligned} V(z = 0) &= \frac{(0)^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right)(0) \\ &= 0 + 0 + 0 \\ &= 0 \quad \checkmark \end{aligned}$$

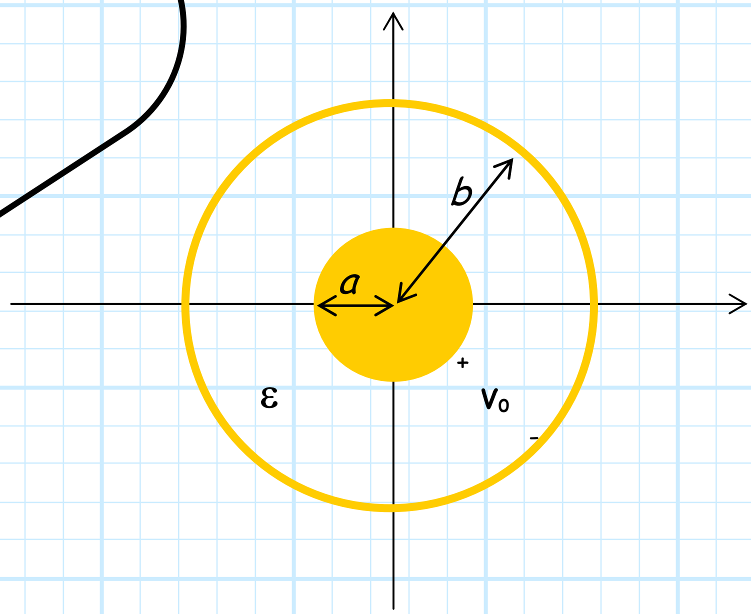
From this result, we can determine the **electric field** $\mathbf{E}(\bar{r})$, the **electric flux density** $\mathbf{D}(\bar{r})$, and the **surface charge density** $\rho_s(\bar{r})$, as before.

Note, however, that the permittivity of the material between the plates is ε_0, as the "dielectric" between the plates is **free-space**.

Example: The Electrostatic Fields of a Coaxial Line

A common form of a transmission line is the **coaxial** cable.





Coax Cross-Section

The coax has an **outer** diameter *b*, and an **inner** diameter *a*. The space between the conductors is filled with **dielectric** material of permittivity ε.

Say a voltage V_0 is placed across the conductors, such that the electric potential of the **outer** conductor is **zero**, and the electric potential of the **inner** conductor is V_0.

The potential **difference** between the inner and outer conductor is therefore $V_0 - 0 = V_0$ volts.

Q: *What electric potential field $V(\bar{r})$, electric field $\mathbf{E}(\bar{r})$ and charge density $\rho_s(\bar{r})$ is produced by this situation?*

A: We must solve a **boundary-value** problem! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Yikes! Where do we start ?

We might start with the electric potential field $V(\bar{r})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's** equation:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon}$$

b) It must also satisfy the **boundary conditions**:

$$V(\rho = a) = V_0 \qquad\qquad V(\rho = b) = 0$$

Consider first the **dielectric** region ($a < \rho < b$). Since the region is a dielectric, there is **no** free charge, and:

$$\rho_v(\bar{\mathbf{r}}) = 0$$

Therefore, Poisson's equation reduces to **Laplace's** equation:

$$\nabla^2 V(\bar{\mathbf{r}}) = 0$$

This particular problem (i.e., coaxial line) is directly solvable because the structure is **cylindrically symmetric**. Rotating the coax around the z-axis (i.e., in the $\hat{a}_\phi$ direction) does not change the geometry at all. As a result, we know that the electric potential field is a function of ρ **only** ! I.E.,:

$$V(\bar{\mathbf{r}}) = V(\rho)$$

This make the problem much **easier**. Laplace's equation becomes:

$$\nabla^2 V(\bar{\mathbf{r}}) = 0$$

$$\nabla^2 V(\rho) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) + 0 + 0 = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) = 0$$

Integrating **both sides** of the resulting equation, we find:

$$\int \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) d\rho = \int 0 \, d\rho$$

$$\rho \frac{\partial V(\rho)}{\partial \rho} = C_1$$

where C_1 is some constant.

Rearranging the above equation, we find:

$$\frac{\partial V(\rho)}{\partial \rho} = \frac{C_1}{\rho}$$

Integrating both sides **again**, we get:

$$\int \frac{\partial V(\rho)}{\partial p} d\rho = \int \frac{C_1}{\rho} \, d\rho$$

$$V(\rho) = C_1 \ln[\rho] + C_2$$

We find that this final equation ($V(\rho) = C_1 \ln[\rho] + C_2$) will satisfy Laplace's equation (try it!).

We must now apply the **boundary conditions** to determine the value of constants C_1 and C_2.

* We know that on the outer surface of the inner conductor (i.e., $\rho = a$), the electric potential is equal to V_0 (i.e., $V(\rho = a) = V_0$).

* And, we know that on the inner surface of the outer conductor (i.e., $\rho = b$) the electric potential is equal to zero (i.e., $V(\rho = b) = 0$).

Therefore, we can write:

$$V(\rho = a) = C_1 \ln[a] + C_2 = V_0$$

$$V(\rho = b) = C_1 \ln[b] + C_2 = 0$$

Two equations and **two** unknowns (C_1 and C_2)!

Solving for C_1 and C_2 we get:

$$C_1 = \frac{-V_0}{\ln[b] - \ln[a]} = \frac{-V_0}{\ln[b/a]}$$

$$C_2 = \frac{V_0 \ln[b]}{\ln[b/a]}$$

and therefore, the **electric potential** field within the dielectric is found to be:

$$V(\bar{r}) = \frac{-V_0 \ln[\rho]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]} \qquad (b > \rho > a)$$

Before we move on, we should do a **sanity check** to make sure we have done everything correctly. Evaluating our result at $\rho = a$, we get:

$$\begin{aligned} V(\rho = a) &= \frac{-V_0 \ln[a]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]} \\ &= \frac{V_0 (\ln[b] - \ln[a])}{\ln[b/a]} \\ &= \frac{V_0 (\ln[b/a])}{\ln[b/a]} \\ &= V_0 \end{aligned}$$

Likewise, we evaluate our result at $\rho = b$:

$$\begin{aligned} V(\rho = b) &= \frac{-V_0 \ln[b]}{\ln[b/a]} + \frac{V_0 \ln[b]}{\ln[b/a]} \\ &= \frac{V_0 (\ln[b] - \ln[b])}{\ln[b/a]} \\ &= 0 \end{aligned}$$

Our result **is** correct!

Now, we can determine the **electric field** within the dielectric by taking the gradient of the electric potential field:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r}) = \frac{V_0}{\ln[b/a]} \frac{1}{\rho} \hat{a}_\rho \quad (b > \rho > a)$$

Note that **electric flux density** is therefore:

$$\mathbf{D}(\bar{\mathbf{r}}) = \varepsilon \mathbf{E}(\bar{\mathbf{r}}) = \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{\rho} \hat{a}_\rho \qquad (b > \rho > a)$$

Finally, we need to determine the **charge density** that actually created these fields!

Q1: *Just where **is** this charge? After all, the dielectric (if it is perfect) will contain **no** free charge.*

A1: The free charge, as we might expect, is in the **conductors**. Specifically, the charge is located at the surface of the conductor.

Q2: *Just how do we **determine** this surface charge $\rho_s(\bar{\mathbf{r}})$?*

A2: Apply the boundary conditions!

Recall that we found that **at** a conductor/dielectric **interface**, the **surface charge density** on the conductor is related to the **electric flux density** in the dielectric as:

$$D_n = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}}) = \rho_s(\bar{\mathbf{r}})$$

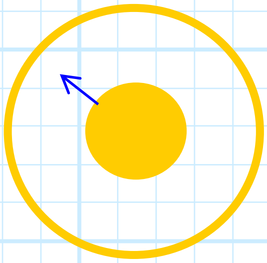
First, we find that the electric flux density **on the surface** of the inner conductor (i.e., at $\rho = a$) is:

$$\mathbf{D}(\bar{\mathbf{r}})\Big|_{\rho=a} = \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{\rho}\Bigg|_{\rho=a}$$

$$= \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a}$$

For **every** point on **outer** surface of the **inner** conductor, we find that the unit vector **normal** to the conductor is:

$$\hat{a}_n = \hat{a}_\rho$$



Therefore, we find that the **surface charge density** on the outer surface of the inner conductor is:

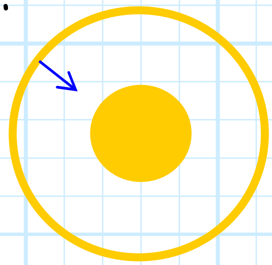
$$\rho_{sa}(\bar{\mathbf{r}}) = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}})\Big|_{\rho=a}$$

$$= \hat{a}_\rho \cdot \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a}$$

$$= \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \qquad (\rho = a)$$

Likewise, we find the unit vector **normal** to the **inner** surface of the **outer** conductor is (do you see why?):

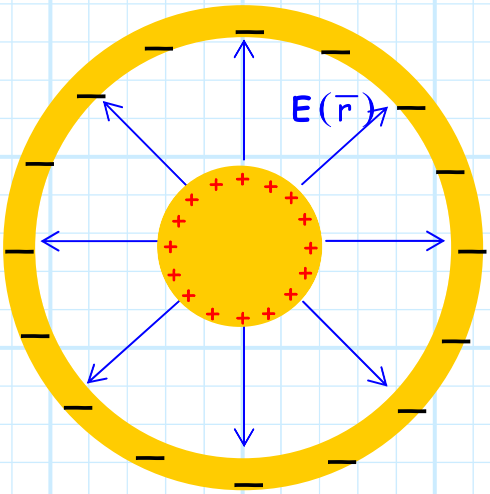
$$\hat{a}_n = -\hat{a}_\rho$$



Therefore, evaluating the electric flux density on the inner surface of the outer conductor (i.e., $\rho = b$), we find:

$$\begin{aligned}\rho_{sb}(\bar{r}) &= \hat{a}_n \cdot \mathbf{D}(\bar{r})\Big|_{\rho=b} \\ &= -\hat{a}_\rho \cdot \hat{a}_\rho \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{b} \\ &= \frac{-\varepsilon V_0}{\ln[b/a]}\frac{1}{b} \qquad (\rho = b)\end{aligned}$$

Note the charge on the outer conductor is **negative**, while that of the inner conductor is **positive**. Hence, the electric field points from the inner conductor to the outer.



We should **note** several things about these solutions:

1) $\nabla \times \mathbf{E}(\bar{r}) = 0$

2) $\nabla \cdot \mathbf{D}(\bar{r}) = 0$ and $\nabla^2 V(\bar{r}) = 0$

3) $\mathbf{D}(\bar{r})$ and $\mathbf{E}(\bar{r})$ are **normal** to the surface of the conductor (i.e., their **tangential** components are equal to **zero**).

4) The **electric field** is precisely the **same** as that given by eq. 4.31 in section 4-5!

$$\mathbf{E}(\bar{r}) = \frac{a\rho_{sa}}{\varepsilon\,\rho}\hat{a}_\rho = \frac{V_0}{\ln[b/a]}\frac{1}{\rho}\hat{a}_\rho \quad (b > \rho > a)$$

In other words, the **fields** $\mathbf{E}(\bar{r})$, $\mathbf{D}(\bar{r})$, and $V(\bar{r})$ are attributable to **free charge densities** $\rho_{sa}(\bar{r})$ and $\rho_{sb}(\bar{r})$.