Solutions to Ampere's Law

Say we **know** the **current distribution** $\mathbf{J}(\bar{r})$ occurring in some physical problem, and we wish to **find** the resulting **magnetic flux density** $\mathbf{B}(\bar{r})$.

Q: *How do we find* $\mathbf{B}(\bar{r})$ *given* $\mathbf{J}(\bar{r})$*?*

A: **Two** ways! We either **directly** solve the differential equation:

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

Or we first solve **this** differential equation for vector field $\mathbf{A}(\bar{r})$:

$$-\nabla^2 \mathbf{A}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

and **then** find $\mathbf{B}(\bar{r})$ by taking the **curl** of $\mathbf{A}(\bar{r})$ (i.e., $\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$).

It turns out that the **second** option is often the easiest!

To see why, consider the **vector Laplacian** operator if vector field $\mathbf{A}(\bar{r})$ is expressed using **Cartesian** base vectors:

$$\nabla^2 \mathbf{A}(\bar{r}) = \nabla^2 A_x(\bar{r})\, \hat{a}_x + \nabla^2 A_y(\bar{r})\, \hat{a}_y + \nabla^2 A_z(\bar{r})\, \hat{a}_z$$

We therefore write **Ampere's Law** in terms of **three** separate **scalar** differential equations:

$$\nabla^2 A_x(\bar{r}) = -\mu_0 J_x(\bar{r})$$

$$\nabla^2 A_y(\bar{r}) = -\mu_0 J_y(\bar{r})$$

$$\nabla^2 A_z(\bar{r}) = -\mu_0 J_z(\bar{r})$$

Each of these differential equations is **easily solved**. In fact, we **already know** their solution!

Recall we had the **exact** same differential equation in electrostatcs (i.e., Poisson's equation):

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

We **know** the solution $V(\bar{r})$ to **this** differential equation is:

$$V(\bar{r}) = \frac{1}{4\pi\,\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

Mathematically, Poisson's equation is **exactly** the same as **each** of the three scalar differential equations at the top of the page, with these **substitutions**:

$$V(\bar{r}) \rightarrow A_x(\bar{r}) \qquad \rho_v(\bar{r}) \rightarrow J_x(\bar{r}) \qquad \frac{1}{\varepsilon_0} \rightarrow \mu_0$$

The **solutions** to the **magnetic** differential equation are therefore:

$$A_x(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_x(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

$$A_y(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_y(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

$$A_z(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_z(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

and since:

$$\mathbf{A}(\bar{r}) = A_x(\bar{r})\ \hat{a}_x + A_y(\bar{r})\ \hat{a}_y + A_z(\bar{r})\ \hat{a}_z$$

and:

$$\mathbf{J}(\bar{r}) = J_x(\bar{r})\ \hat{a}_x + J_y(\bar{r})\ \hat{a}_y + J_z(\bar{r})\ \hat{a}_z$$

we can **combine** these three solutions and get the **vector** solution to our **vector** differential equation:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

Therefore, given **current distribution** $\mathbf{J}(\bar{r})$, we use the above equation to determine **magnetic vector potential** $\mathbf{A}(\bar{r})$. We **then** take the **curl** of this result to determine **magnetic flux density** $\mathbf{B}(\bar{r})$.

For surface current, the resulting magnetic vector potential is:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} ds'$$

and for a current I flowing along contour C, we find:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0 \, I}{4\pi} \oint_C \frac{\overline{d\ell'}}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$

Again, ponder the **analogy** between these equations involving **sources** and **potentials** and the equivalent equation from **electrostatics**:

$$V(\bar{\mathbf{r}}) = \frac{1}{4\pi \varepsilon_0} \iiint_V \frac{\rho_v(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} dv'$$