Spherical base vectors are the “natural” base vectors of a sphere.

\( \hat{a}_r \) points in the direction of increasing \( r \). In other words \( \hat{a}_r \) points away from the origin. This is analogous to the direction we call up.

\( \hat{a}_\phi \) points in the direction of increasing \( \phi \). This is analogous to the direction we call south.

\( \hat{a}_\theta \) points in the direction of increasing \( \theta \). This is analogous to the direction we call east.
IMPORTANT NOTE: The directions of spherical base vectors are dependent on position. First you must determine where you are in space (using coordinate values), then you can define the directions of \( \hat{a}_r, \hat{a}_\theta, \hat{a}_\phi \).

Note Cartesian base vectors are special, in that their directions are independent of location—they have the same directions throughout all space.

Thus, it is helpful to define spherical base vectors in terms of Cartesian base vectors. It can be shown that:

\[
\begin{align*}
\hat{a}_r \cdot \hat{a}_x &= \sin \theta \cos \phi \\
\hat{a}_\theta \cdot \hat{a}_x &= \cos \theta \cos \phi \\
\hat{a}_\phi \cdot \hat{a}_x &= -\sin \phi \\
\hat{a}_r \cdot \hat{a}_y &= \sin \theta \sin \phi \\
\hat{a}_\theta \cdot \hat{a}_y &= \cos \theta \sin \phi \\
\hat{a}_\phi \cdot \hat{a}_y &= \cos \phi \\
\hat{a}_r \cdot \hat{a}_z &= \cos \theta \\
\hat{a}_\theta \cdot \hat{a}_z &= -\sin \theta \\
\hat{a}_\phi \cdot \hat{a}_z &= 0
\end{align*}
\]

Recall that any vector \( \mathbf{A} \) can be written as:

\[
\mathbf{A} = (\mathbf{A} \cdot \hat{a}_x) \hat{a}_x + (\mathbf{A} \cdot \hat{a}_y) \hat{a}_y + (\mathbf{A} \cdot \hat{a}_z) \hat{a}_z.
\]

Therefore, we can write unit vector \( \hat{a}_r \) as, for example:

\[
\hat{a}_r = (\hat{a}_r \cdot \hat{a}_x) \hat{a}_x + (\hat{a}_r \cdot \hat{a}_y) \hat{a}_y + (\hat{a}_r \cdot \hat{a}_z) \hat{a}_z
\]

\[
= \sin \theta \cos \phi \ \hat{a}_x + \sin \theta \sin \phi \ \hat{a}_y + \cos \theta \ \hat{a}_z
\]

This result explicitly shows that \( \hat{a}_r \) is a function of \( \theta \) and \( \phi \).
For example, at the point in space \( r = 7.239 \), \( \theta = 90^\circ \) and \( \phi = 0^\circ \), we find that \( \hat{\mathbf{a}}_r = \hat{\mathbf{a}}_x \). In other words, at this point in space, the direction \( \hat{\mathbf{a}}_r \) points in the \( x \)-direction.

Or, at the point in space \( r = 2.735 \), \( \theta = 90^\circ \) and \( \phi = 90^\circ \), we find that \( \hat{\mathbf{a}}_r = \hat{\mathbf{a}}_y \). In other words, at this point in space, \( \hat{\mathbf{a}}_r \) points in the \( y \)-direction.

Additionally, we can write \( \hat{\mathbf{a}}_\theta \) and \( \hat{\mathbf{a}}_\phi \) as:

\[
\hat{a}_\theta = (\hat{a}_\theta \cdot \hat{a}_x) \hat{a}_x + (\hat{a}_\theta \cdot \hat{a}_y) \hat{a}_y + (\hat{a}_\theta \cdot \hat{a}_z) \hat{a}_z
\]

\[
\hat{a}_\phi = (\hat{a}_\phi \cdot \hat{a}_x) \hat{a}_x + (\hat{a}_\phi \cdot \hat{a}_y) \hat{a}_y + (\hat{a}_\phi \cdot \hat{a}_z) \hat{a}_z
\]

Alternatively, we can write **Cartesian** base vectors in terms of spherical base vectors, i.e.,

\[
\hat{a}_x = (\hat{a}_x \cdot \hat{a}_r) \hat{a}_r + (\hat{a}_x \cdot \hat{a}_\theta) \hat{a}_\theta + (\hat{a}_x \cdot \hat{a}_\phi) \hat{a}_\phi
\]

\[
\hat{a}_y = (\hat{a}_y \cdot \hat{a}_r) \hat{a}_r + (\hat{a}_y \cdot \hat{a}_\theta) \hat{a}_\theta + (\hat{a}_y \cdot \hat{a}_\phi) \hat{a}_\phi
\]

\[
\hat{a}_z = (\hat{a}_z \cdot \hat{a}_r) \hat{a}_r + (\hat{a}_z \cdot \hat{a}_\theta) \hat{a}_\theta + (\hat{a}_z \cdot \hat{a}_\phi) \hat{a}_\phi
\]

Using the table on the previous page, we can insert the result of each dot product to express each base vector in terms of spherical coordinates!