Spherically Symmetric Charge Densities

Consider volume charge densities $\rho_v(\bar{\mathrm{r}})$ that are functions of spherical coordinate r **only**, e.g.:

$$\rho_v(\bar{\mathrm{r}}) = \frac{1}{r^2} \qquad \text{or} \qquad \rho_v(\bar{\mathrm{r}}) = e^{-r}$$

We call these types of charge densities **spherically symmetric**, as the charge density changes as a function of the distance from the origin only (i.e., is independent of coordinates θ or ϕ).

As a result, the charge distribution in this case looks sort of like a "**fuzzy ball**", centered at the origin!

Using the point form of Gauss's Law, we find the resulting static electric field **must** have the form:

$$\mathbf{E}(\bar{\mathrm{r}}) = E(r)\,\hat{a}_r \quad \text{(for spherically symmetric } \rho_v(\bar{\mathrm{r}}) \text{)}$$

Think about what **this** says. It states that the resulting static electric field from a spherically symmetric charge density is:

* A function of spherical coordinate r **only.**

* Points in the direction $\hat{a}_r$ (i.e., away from the origin at every point).

As a result, we can use the **integral form** of Gauss's Law to determine the **specific scalar** function $E(r)$ resulting from some **specific**, spherically symmetric charge density $\rho_v(\bar{\mathrm{r}})$.

Recall the integral form of **Gauss's Law**:

$$\oint\!\!\oint_S \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{\mathrm{r}})\, dv$$

Consider now a surface S that is a **sphere** with radius r, centered at the origin. We call this surface the **Gaussian Surface** for spherically symmetric charge densities.

To we why, we integrate over this Gaussian surface and find:

$$\oint\!\!\oint_S \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{ds} = \int_0^{2\pi}\int_0^{\pi} \mathbf{E}(\bar{\mathrm{r}}) \cdot \hat{a}_r \; r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} E(r)\, \hat{a}_r \cdot \hat{a}_r \; r^2 \sin\theta \, d\theta \, d\phi$$

$$= E(r)\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= 4\pi r^2 E(r)$$

Therefore, from Gauss's Law, we get:

$$4\pi r^2 E(r) = \frac{Q_{enc}}{\varepsilon_0}$$

Rearranging, we find that the function E(*r*) is:

$$E(r) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$$

The enclosed charge Q_{enc} can be determined for a **spherically symmetric** distribution (a function of *r* only!) as:

$$\begin{aligned} Q_{enc} &= \iiint_V \rho_v(\bar{\mathrm{r}})\, dv \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_v(r')\, r'^2 \sin\theta \, dr' d\theta d\phi \\ &= 4\pi \int_0^{r} \rho_v(r')\, r'^2 dr' \end{aligned}$$

Therefore, we find that the static electric field produced by a **spherically symmetric** charge density is $\mathbf{E}(\bar{\mathrm{r}}) = E(\mathrm{r})\hat{a}_r$, where the scalar function E(*r*) is:

$$\begin{aligned} E(r) &= \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \\ &= \frac{1}{\varepsilon_0 r^2} \int_0^{r} \rho_v(r')\; r'^2 dr' \end{aligned}$$

Or, more specifically, we find that the static electric field produced by some **spherically symmetric** charge density $\rho_v(\bar{r})$ is:

$$\begin{aligned}\mathbf{E}(\bar{r}) &= \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{a}_r \\ &= \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\, r'^2 dr'\end{aligned}$$

Thus, for a **spherically symmetric** charge density, we can find the resulting electric field **without** the difficult integration and evaluation required by **Coulomb's Law**!