The Biot-Savart Law

So, we now know that given some current density, we can find the resulting magnetic vector potential \( A(\mathbf{r}) \):

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV'
\]

and then determine the resulting magnetic flux density \( B(\mathbf{r}) \) by taking the curl:

\[
B(\mathbf{r}) = \nabla \times A(\mathbf{r})
\]

Q: Golly, can’t we somehow combine the curl operation and the magnetic vector potential integral?

A: Yes! The result is known as the Biot-Savart Law.

Combining the two above equations, we get:

\[
B(\mathbf{r}) = \nabla \times \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV'
\]

This result is of course not very helpful, but we note that we can move the curl operation into the integrand:
\[ B(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \nabla \times \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\nu' \]

Note this result reverses the process: first we perform the curl, and then we integrate.

We can do this is because the integral is over the primed coordinates (i.e., \(\vec{r}'\)) that specify the sources (current density), while the curl take the derivatives of the unprimed coordinates (i.e., \(\vec{r}\)) that describe the fields (magnetic flux density).

**Q:** Yikes! That curl operation still looks particularly difficult. How we perform it?

**A:** We take advantage of a know vector identity! The curl of vector field \(f(\vec{r})G(\vec{r})\), where \(f(\vec{r})\) is any scalar field and \(G(\vec{r})\) is any vector field, can be evaluated as:

\[ \nabla \times (f(\vec{r})G(\vec{r})) = f(\vec{r})\nabla \times G(\vec{r}) - G(\vec{r}) \times \nabla f(\vec{r}) \]

Note the integrand of the above equation is in the form \(\nabla \times (f(\vec{r})G(\vec{r}))\), where:

\[ f(\vec{r}) = \frac{1}{|\vec{r} - \vec{r}'|} \quad \text{and} \quad G(\vec{r}) = J(\vec{r}') \]

Therefore we find:
\[ \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \times \mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}') \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \]

In the first term we take the curl of \( \mathbf{J}(\mathbf{r}') \). Note however that this vector field is a constant with respect to the unprimed coordinates \( \mathbf{r} \). Thus the derivatives in the curl will all be equal to zero, and we find that:

\[ \nabla \times \mathbf{J}(\mathbf{r}') = 0 \]

Likewise, it can be shown that:

\[ \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \]

Using these results, we find:

\[ \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

and therefore the magnetic flux density is:

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \int \int_{V'} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \]

This is known as the **Biot-Savart Law**!
For a **surface** current $\mathbf{J}_s(\mathbf{r})$, the Biot-Savart Law becomes:

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_S \mathbf{J}_s(\mathbf{r'}) \times \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|^3} d\mathbf{s'}
$$

and for **line** current $I$, flowing on contour $C$, the Biot-Savart Law is:

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l'} \times \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|^3}
$$

Note the contour $C$ is **closed**. Do **you** know why?

This is **dad-gum** outstanding! The Biot-Savart Law allows us to **directly** determine magnetic flux density $\mathbf{B}(\mathbf{r})$, given some current density $\mathbf{J}(\mathbf{r})$!

Note that the Biot-Savart Law is therefore **analogous** to Coulomb's Law in Electrostatics (Do you see why?)!