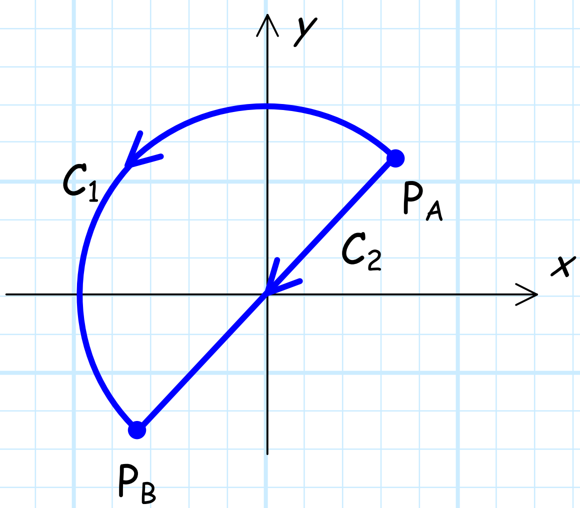
The Conservative Vector Field

Of all possible vector fields $\mathbf{A}(\bar{r})$, there is a subset of vector fields called **conservative** fields. A conservative vector field is a vector field that can be expressed as the **gradient** of some scalar field $g(\bar{r})$:

$$\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$$

In other words, the gradient of **any** scalar field **always** results in a conservative field !

As we discussed earlier, a conservative field has the interesting property that its line integral is dependent on the **beginning** and **ending** points of the contour **only**! In other words, for the two contours:



we find that:

$$\int_{C_1} \mathbf{C}(\bar{r}) \cdot \overline{d\ell} = \int_{C_2} \mathbf{C}(\bar{r}) \cdot \overline{d\ell}$$

We therefore say that the line integral of a conservative field **is path independent**.

This path independence is evident when considering the **integral identity**:

$$\int_C \nabla g(\bar{r}) \cdot \overline{d\ell} = g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A)$$

where position vector $\bar{r}_B$ denotes the **ending** point (P_B) of contour C, and $\bar{r}_A$ denotes the **beginning** point (P_A). Likewise, $g(\bar{r} = \bar{r}_B)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_B$, and $g(\bar{r} = \bar{r}_A)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_A$.

Note for **one** dimension, the above identity simply reduces to the familiar expression:

$$\int_{x_a}^{x_b} \frac{\partial g(x)}{\partial x} dx = g(x = x_b) - g(x = x_a)$$

Since **every** conservative field can be written in terms of the **gradient** of a scalar field, we can use this identity to conclude:

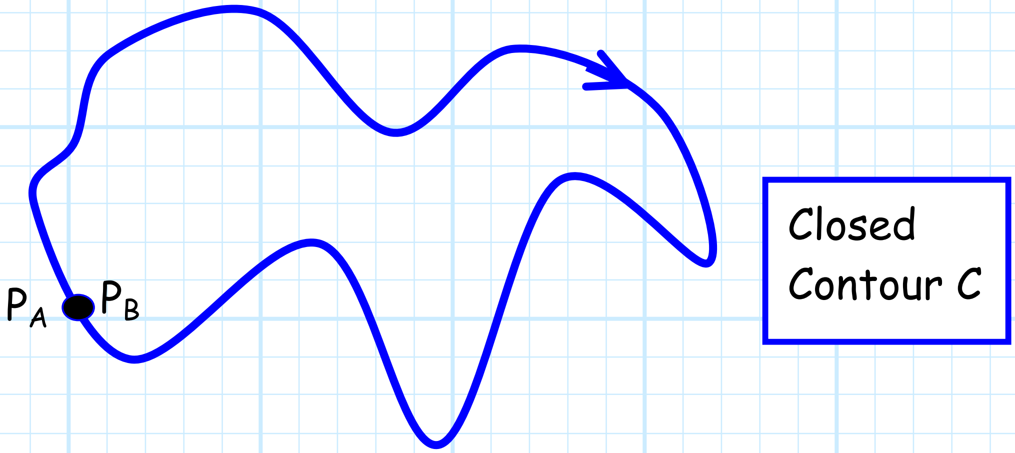
$$\begin{aligned}\int_C \mathbf{C}(\bar{r}) \cdot \overline{d\ell} &= \int_C \nabla g(\bar{r}) \cdot \overline{d\ell} \\ &= g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A)\end{aligned}$$

Thus, the line integral **only** depends on the value $g(\bar{r})$ at the beginning and end points of a contour, the **path** taken to connect these points makes **no** difference!

Consider then what happens then if we integrate over a **closed** contour.

Q: *What the heck is a closed contour ??*

A: A closed contour is a contour whose beginning and ending is the **same** point! E.G.,



* A contour that is **not** closed is refered to as an **open** contour.

* Integration over a closed contour is **denoted** as:

$$\oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

* The integration of a **conservative** field over a **closed** contour is therefore:

$$\begin{aligned}\oint_C \mathbf{C}(\bar{r}) \cdot \overline{d\ell} &= \oint_C \nabla g(\bar{r}) \cdot \overline{d\ell} \\ &= g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A) \\ &= 0\end{aligned}$$

This result is due to the fact that $\bar{r}_A = \bar{r}_B$, therefore;

$$g\left(\bar{r} = \bar{r}_A\right) = g\left(\bar{r} = \bar{r}_B\right)$$

and thus the **subtraction** of these two values is **always zero**!

*Let's **summarize** what we know about a **conservative** vector field:*

1. A conservative vector field can always be expressed as the **gradient** of a **scalar** field.

2. The gradient of **any** scalar field is therefore a conservative vector field.

3. Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).

4. Integration of a conservative vector field over any **closed** contour is always equal to **zero**.