The Continuity Equation

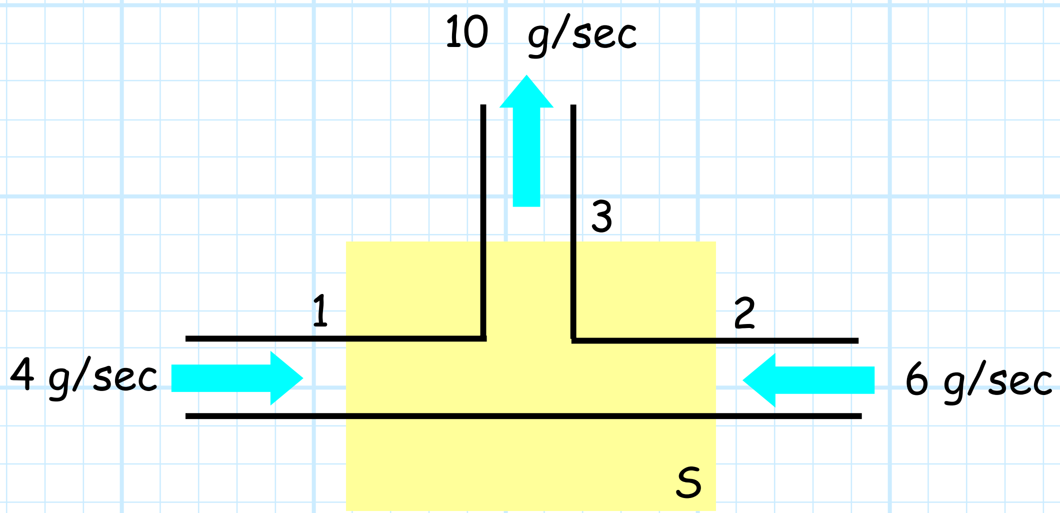
For **some** closed surfaces,

$$I = \oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = \frac{dQ}{dt} \neq 0 \quad !!!$$

Q: *How is this possible ? I thought you said charge cannot be **created** or **destroyed**.*

A: Let's try this **analogy**.

Say we have three pipes that carry **water** to/from a node:



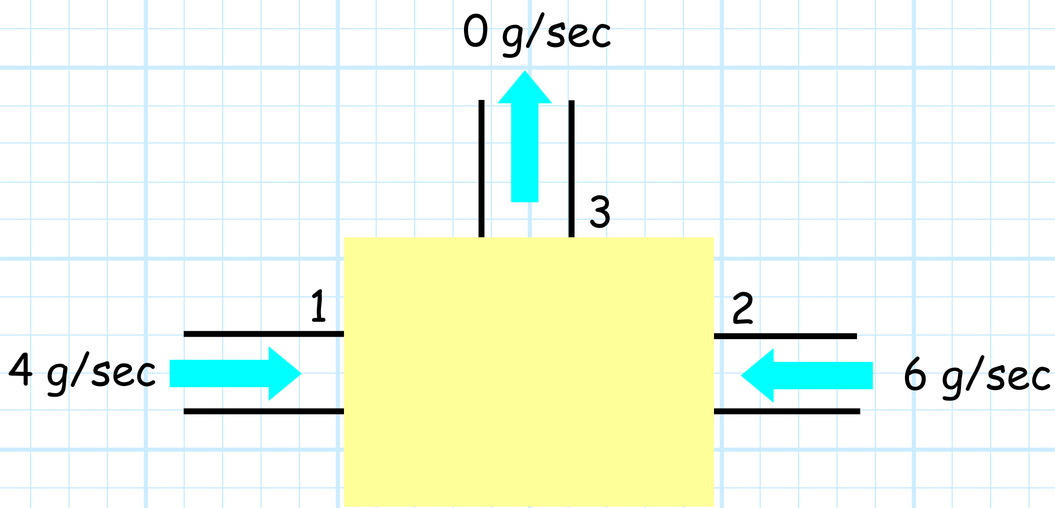
If a current of **4** gallons/second **enters** the node through pipe 1, and another **6** gallons/second **enters** through pipe 2, then **10** gallons/second **must** be **leaving** the node through pipe 3.

The reason for this of course is that water cannot be **created** or **destroyed**, and therefore if water **enters** surface S at a rate of 10 gallons/sec, then water must also **leave** at the same rate.

Therefore, the **amount of water** $W(t)$ in closed surface S remains **constant** with time. I.E.,

$$\frac{d\,W(t)}{dt} = 0$$

Now, consider the system below. Water is entering through pipe 1 and pipe 2, **again** at a rate of 4 gallons/second and 6 gallons/second, respectively.

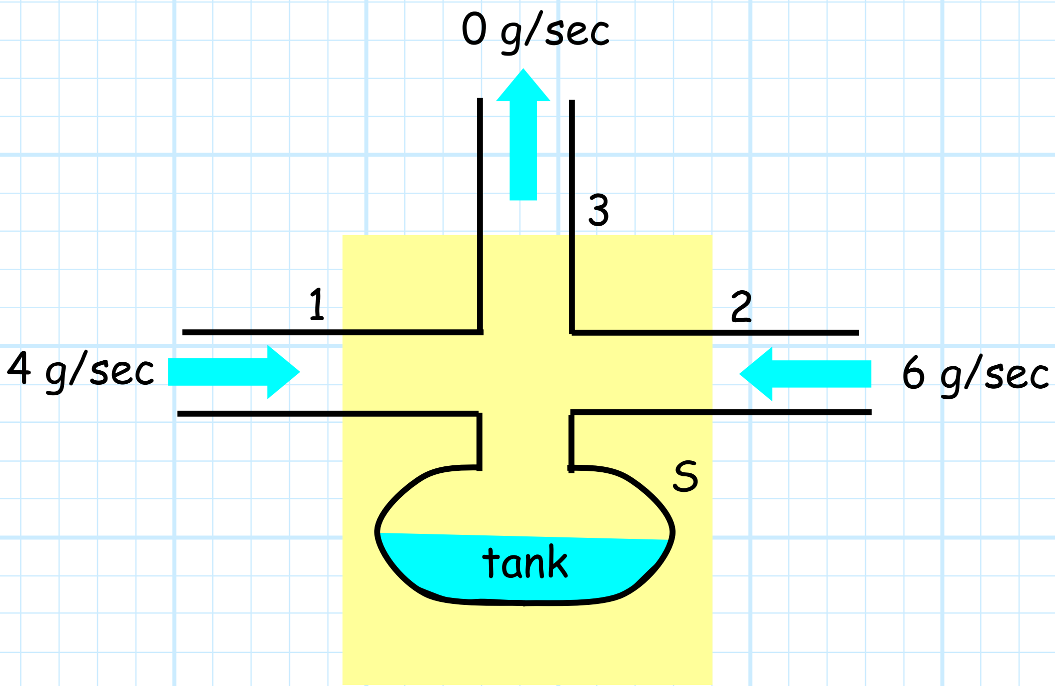


However, this time we find that **no** water is leaving through pipe 3! Therefore:

$$\frac{d\,W(t)}{dt} \neq 0$$

Q: ***How*** *is this possible? What happens to the water?*

A: It's possible because the closed surface S surrounds a **storage device** (i.e., a water **tank**).



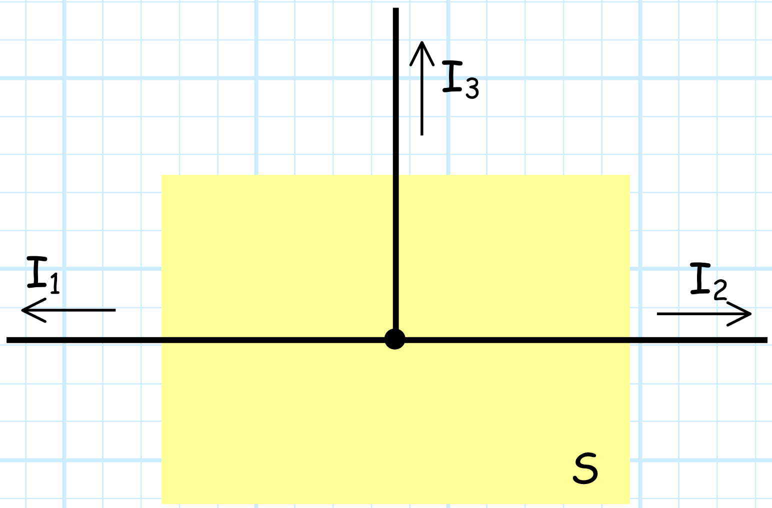
In addition to being a **sink** for water, this tank can also be a **source**. As a result, the current exiting pipe 3 could also **exceed** 10 gallons/second !

The "catch" here is that this **cannot last forever**. Eventually, the tank will get completely **full** or completely **empty**. After that we will find again that $dW(t)/dt = 0$.

Now, let's return to **charge**.

It would likewise **appear** that the charge **enclosed** ($Q_{enc}(t)$) within some surface that surrounds a circuit node must **always** be constant with respect to time. I.E.,

$$\frac{d\,Q_{enc}(t)}{dt} = 0$$



Therefore:

$$I = \oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$$

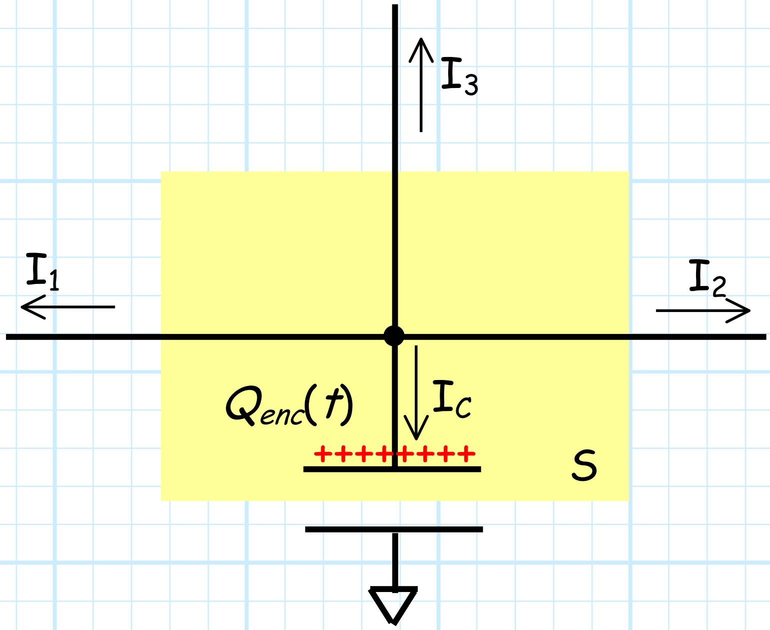
or

$$\sum_{n=1}^{N} I_n = 0$$

But, there is such a thing as a charge "tank"!

A charge tank is a **capacitor**.

A capacitor can either store or source **enclosed** charge $Q_{enc}(t)$, such that $d\,Q_{enc}(t)/dt \neq 0$.



The current I_C is known as **displacement current.** We find that:

$$I_C = -\sum_{n=1}^{N} I_n$$

Meaning of course that KCL must **also** include displacement current:

$$I_C + \sum_{n=1}^{N} I_n = 0$$

Now, recall again that:

$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = \frac{dQ(t)}{dt}$$

where $Q(t)$ represents the charge moving **from** the inside of surface S **to** the outside the surface S.

Note an increase in the charge **outside** the surface S results in a corresponding decrease in the total charge **enclosed** by S (i.e., $Q_{enc}(t)$). Therefore:

$$\frac{d\,Q(t)_{enc}}{dt} = -\frac{d\,Q(t)}{dt}$$

If these derivatives are not zero, then **displacement current** must exist with in volume surrounded by S!

The **value** of this displacement current is equal to $dQ_{enc}(t)/dt$.

Thus, if **displacement** current exists (meaning that there is some way to "store" charge) the **continuity equation** becomes:

$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = -\frac{dQ_{enc}(t)}{dt} = -I_C$$

Note this means that the current flowing **out** of surface S (i.e., I) is equal to the **opposite** value of displacement current $dQ_{enc}(t)/dt$.

This of course means that the current **entering** surface S (i.e., -I) is **equal** to the displacement current $dQ_{enc}(t)/dt$.

Makes sense! If the total current flowing **into** a closed surface S is **positive**, then the total charge enclosed by the surface is **increasing.** This charge must all be stored somewhere, as it cannot be destroyed!

The continuity equation can therefore alternatively be written as:

$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} + \frac{dQ_{enc}(t)}{dt} = 0$$

$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} + I_c = 0$$

If displacement current does **not** exist, then $d\,Q_{enc}(t)/dt = 0$ and the continuity equation remains:

$$I = \oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$$