The Contour C

In this class, we will limit ourselves to studying only those contours that are formed when we change the location of a point by varying just one coordinate parameter. In other words, the other two coordinate parameters will remain fixed.

Mathematically, therefore, a contour is described by:

2 equalities (e.g., \( x = 2, y = -4; \ r = 3, \phi = \pi/4 \))

AND

1 inequality (e.g., \(-1 < z < 5; \ 0 < \theta < \pi/2 \))

Likewise, we will need to explicitly determine the differential displacement vector \( \overrightarrow{d\ell} \) for each contour.

Recall we have studied seven coordinate parameters \((x, y, z, \rho, \phi, r, \theta)\). As a result, we can form seven different contours \( \mathcal{C} \).

**Cartesian Contours**

Say we move a point from \( P(x = 1, y = 2, z = -3) \) to \( P(x = 1, y = 2, z = 3) \) by changing only the coordinate variable \( z \) from \( z = -3 \) to \( z = 3 \). In other words, the coordinate values \( x \) and \( y \) remain constant at \( x = 1 \) and \( y = 2 \).
We form a contour that is a line segment, parallel to the z-axis!

Note that every point along this segment has coordinate values \( x = 1 \) and \( y = 2 \). As we move along the contour, the only coordinate value that changes is \( z \).

Therefore, the differential directed distance associated with a change in position from \( z \) to \( z + dz \), is \( \overrightarrow{d\ell} = dz = \hat{a}_z dz \).
Similarly, a line segment parallel to the $x$-axis (or $y$-axis) can be formed by changing coordinate parameter $x$ (or $y$), with a resulting differential displacement vector of $d\ell = dx = \hat{a}_x dx$ (or $d\ell = dy = \hat{a}_y dy$).

The three Cartesian contours are therefore:

1. **Line segment parallel to the $z$-axis**

   \[
   x = c_x \quad y = c_y \quad c_{z1} \leq z \leq c_{z2}
   \]

   \[
   d\ell = \hat{a}_z dz
   \]

2. **Line segment parallel to the $y$-axis**

   \[
   x = c_x \quad c_{y1} \leq y \leq c_{y2} \quad z = c_z
   \]

   \[
   d\ell = \hat{a}_y dy
   \]

3. **Line segment parallel to the $x$-axis**

   \[
   c_{x1} \leq x \leq c_{x2} \quad y = c_y \quad z = c_z
   \]

   \[
   d\ell = \hat{a}_x dx
   \]
**Cylindrical Contours**

Say we move a point from \(P(\rho=1, \phi=45^\circ, z=2)\) to \(P(\rho=3, \phi=45^\circ, z=2)\) by changing only the coordinate variable \(\rho\) from \(\rho=1\) to \(\rho=3\). In other words, the coordinate values \(\phi\) and \(z\) remain constant at \(\phi=45^\circ\) and \(z=2\).

We form a contour that is a line segment, parallel to the \(x-y\) plane (i.e., perpendicular to the \(z\)-axis).

Note that every point along this segment has coordinate values \(\phi=45^\circ\) and \(z=2\). As we move along the contour, the only coordinate value that changes is \(\rho\).
Therefore, the **differential** directed distance associated with a change in position from $\rho$ to $\rho + d\rho$, is $d\ell = d\rho = \hat{a}_\rho \, d\rho$.

Alternatively, say we move a point from $P(\rho=3, \phi=0, z=2)$ to $P(\rho=3, \phi=90^\circ, z=2)$ by changing **only** the coordinate variable $\phi$ from $\phi=0$ to $\phi=90^\circ$. In other words, the coordinate values $\rho$ and $z$ remain **constant** at $\rho=3$ and $z=2$.

We form a contour that is a **circular arc**, parallel to the $x$-$y$ plane.
Note: if we move from $\phi = 0$ to $\phi = 360^\circ$, a complete circle is formed around the z-axis.

Every point along the arc has coordinate values $\rho = 3$ and $z = 2$. As we move along the contour, the only coordinate value that changes is $\phi$.

Therefore, the differential directed distance associated with a change in position from $\phi$ to $\phi + d\phi$, is:

$$d\ell = d\phi = \rho d\phi.$$

Finally, changing coordinate $z$ generates the third cylindrical contour—but we already did that in Cartesian coordinates! The result is again a line segment parallel to the z-axis.

The three cylindrical contours are therefore described as:
1. Line segment parallel to the z-axis.

\[ \rho = c_\rho \quad \phi = c_\phi \quad c_{z1} \leq z \leq c_{z2} \]

\[ \overline{d\ell} = \hat{a}_z \ dz \]

2. Circular arc parallel to the x-y plane.

\[ \rho = c_\rho \quad c_{\phi1} \leq \phi \leq c_{\phi2} \quad z = c_z \]

\[ \overline{d\ell} = \hat{a}_\phi \ \rho d\phi \]

3. Line segment parallel to the x-y plane.

\[ c_{\rho1} \leq \rho \leq c_{\rho2} \quad \phi = c_\phi \quad z = c_z \]

\[ \overline{d\ell} = \hat{a}_\rho \ d\rho \]
Spherical Contours

Say we move a point from \( P(r =0, \theta = 60^\circ, \phi = 45^\circ) \) to \( P(r =3, \theta = 60^\circ, \phi = 45^\circ) \) by changing only the coordinate variable \( r \) from \( r = 0 \) to \( r = 3 \). In other words, the coordinate values \( \theta \) and \( \phi \) remain constant at \( \theta = 60^\circ \) and \( \phi = 45^\circ \).

We form a contour that is a line segment, emerging from the origin.

Every point along the line segment has coordinate values \( \theta = 60^\circ \) and \( \phi = 45^\circ \). As we move along the contour, the only coordinate value that changes is \( r \).
Therefore, the **differential** directed distance associated with a change in position from \( r \) to \( r + dr \), is \( \overrightarrow{d\ell} = \overrightarrow{dr} = \hat{a}_r \, dr \).

Alternatively, say we move a point from \( P(r=3, \theta=0, \phi=45^\circ) \) to \( P(r=3, \theta=90^\circ, \phi=45^\circ) \) by changing **only** the coordinate variable \( \theta \) from \( \theta = 0 \) to \( \theta = 90^\circ \). In other words, the coordinate values \( \theta \) and \( \phi \) remain **constant** at \( \theta = 60^\circ \) and \( \phi = 45^\circ \).

We form a **circular arc**, whose plane includes the \( z \)-axis.
Every point along the arc has coordinate values $r = 3$ and $\phi = 45^\circ$. As we move along the contour, the only coordinate value that changes is $\theta$.

Therefore, the differential directed distance associated with a change in position from $\theta$ to $\theta + d\theta$, is $d\ell = d\theta = \hat{\theta} r d\theta$.

Finally, we could fix coordinates $r$ and $\theta$ and vary coordinate $\phi$ only—but we already did this in cylindrical coordinates! We again find that a circular arc is generated, an arc that is parallel to the $x$-$y$ plane.

The three spherical contours are therefore:
1. A circular arc parallel to the x-y plane.

\[ r = c_r, \quad \theta = c_\theta, \quad c_{\phi 1} \leq \phi \leq c_{\phi 2} \]

\[ \overline{d\ell} = \hat{\phi} \quad r \sin \theta \, d\phi \]

2. A circular arc in a plane that includes the z-axis.

\[ r = c_r, \quad c_\theta 1 \leq \theta \leq c_\theta 2, \quad \phi = c_\phi \]

\[ \overline{d\ell} = \hat{\phi} \quad r \, d\theta \]

3. A line segment directed toward the origin.

\[ c_{r 1} \leq r \leq c_{r 2}, \quad \theta = c_\theta, \quad \phi = c_\phi \]

\[ \overline{d\ell} = \hat{r} \quad dr \]