The Contour C

In this class, we will limit ourselves to studying only those contours that are formed when we change the location of a point by varying **just one** coordinate parameter. In other words, the other two coordinate parameters will remain **fixed**.

Mathematically, therefore, a **contour** is described by:

2 equalities (e.g., $x=2$, $y=-4$; $r=3$, $\phi=\pi/4$)

AND

1 inequality (e.g., $-1<z<5$; $0<\theta<\pi/2$)

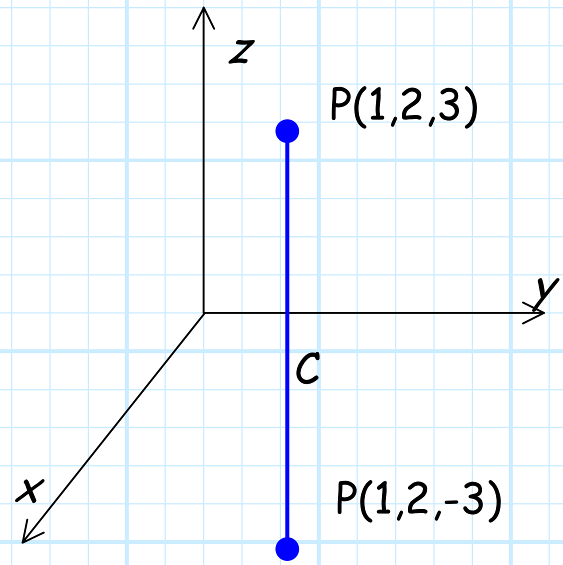
Likewise, we will need to explicitly determine the differential displacement vector $\overline{d\ell}$ for each contour.

Recall we have studied **seven** coordinate parameters ($x, y, z, \rho, \phi, r, \theta$). As a result, we can form **seven** different contours C!

Cartesian Contours

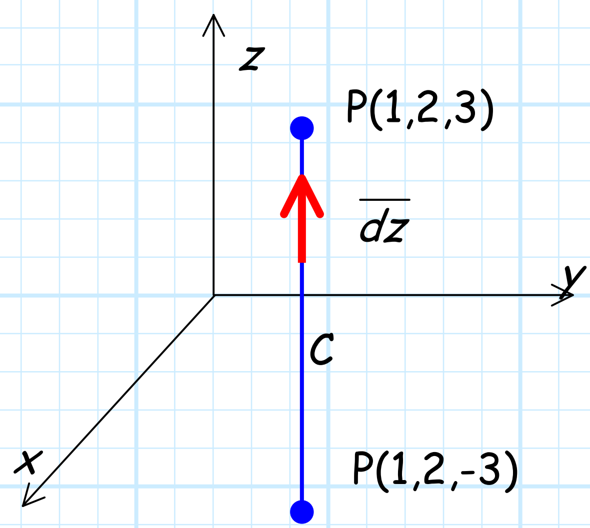
Say we move a point from P($x=1$, $y=2$, $z=-3$) to P($x=1$, $y=2$, $z=3$) by changing **only** the coordinate variable z from $z=-3$ to $z=3$. In other words, the coordinate values x and y remain **constant** at $x=1$ and $y=2$.

We form a contour that is a **line segment**, **parallel** to the *z*-axis!



Note that **every** point along this segment has coordinate values $x=1$ and $y=2$. As we move along the contour, the **only** coordinate value that changes is *z*.

Therefore, the **differential** directed distance associated with a change in position from z to $z+dz$, is $\overline{d\ell} = \overline{dz} = \hat{a}_z dz$.



Similarly, a line segment parallel to the *x*-axis (or *y*-axis) can be formed by changing coordinate parameter x (or y), with a resulting differential displacement vector of $\overline{d\ell} = \overline{dx} = \hat{a}_x dx$ (or $\overline{d\ell} = \overline{dy} = \hat{a}_y dy$).

The three Cartesian contours are therefore:

1. *Line segment parallel to the z-axis*

$$x = c_x \qquad y = c_y \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. *Line segment parallel to the y-axis*

$$x = c_x \qquad c_{y1} \le y \le c_{y2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_y \, dy$$

3. *Line segment parallel to the x-axis*

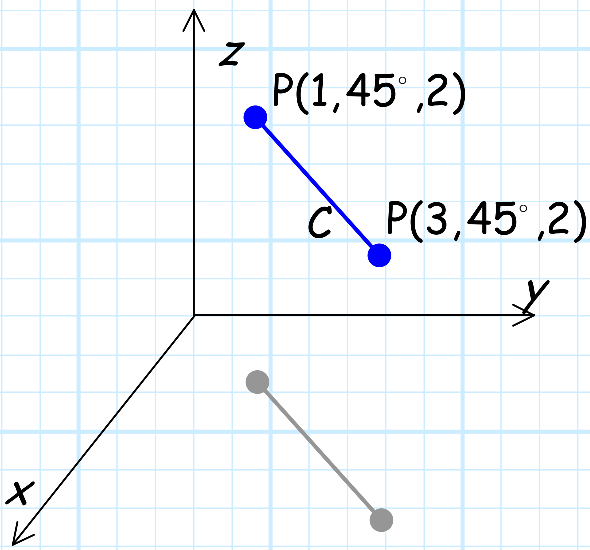
$$c_{x1} \le x \le c_{x2} \qquad y = c_y \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_x \, dx$$

Cylindrical Contours

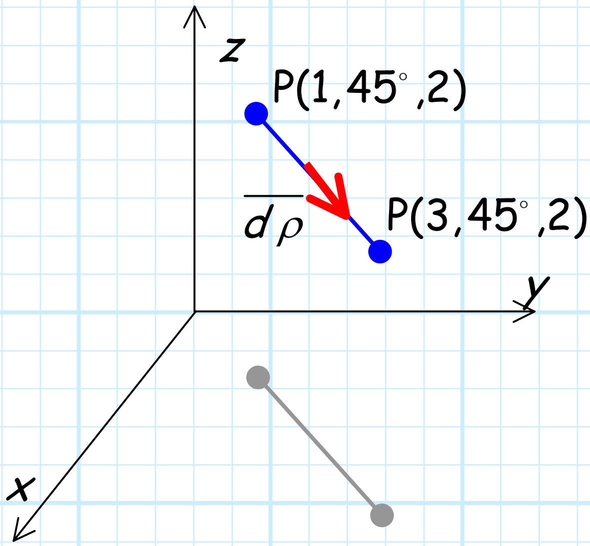
Say we move a point from P(ρ=1, $\phi = 45^\circ$, z =2) to P(ρ=3, $\phi = 45^\circ$, z =2) by changing **only** the coordinate variable ρ from ρ =*1* to ρ=3. In other words, the coordinate values ϕ and z remain **constant** at $\phi = 45^\circ$ and z =2.

We form a contour that is a **line segment**, **parallel** to the *x-y* plane (i.e., perpendicular to the *z*-axis).



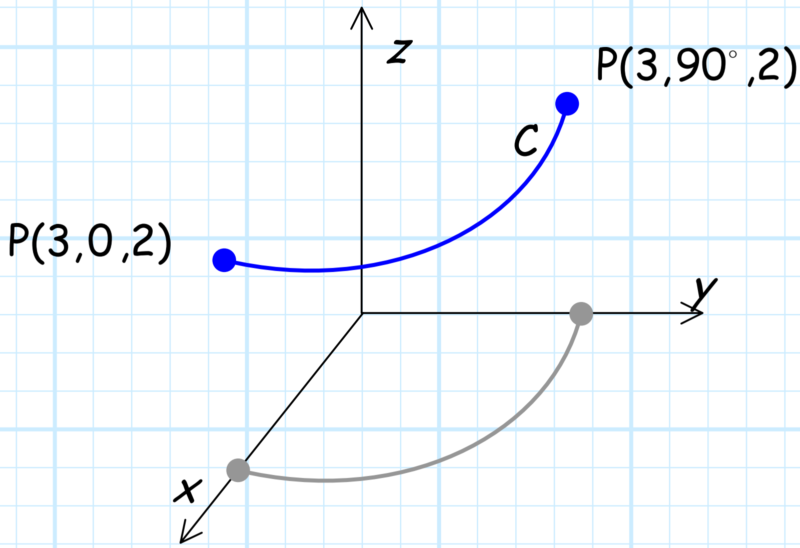
Note that **every** point along this segment has coordinate values $\phi = 45^\circ$ and z =2. As we move along the contour, the **only** coordinate value that changes is ρ.

Therefore, the **differential** directed distance associated with a change in position from ρ to $\rho + d\rho$, is $\overline{d\ell} = \overline{d\rho} = \hat{a}_\rho \, d\rho$.



Alternatively, say we move a point from P(ρ=3, $\phi = 0$, z=2) to P(ρ=3, $\phi = 90^\circ$, z=2) by changing **only** the coordinate variable ϕ from ϕ=0 to $\phi = 90^\circ$. In other words, the coordinate values ρ and z remain **constant** at ρ=3 and z=2.

We form a contour that is a **circular arc**, parallel to the x-y plane.

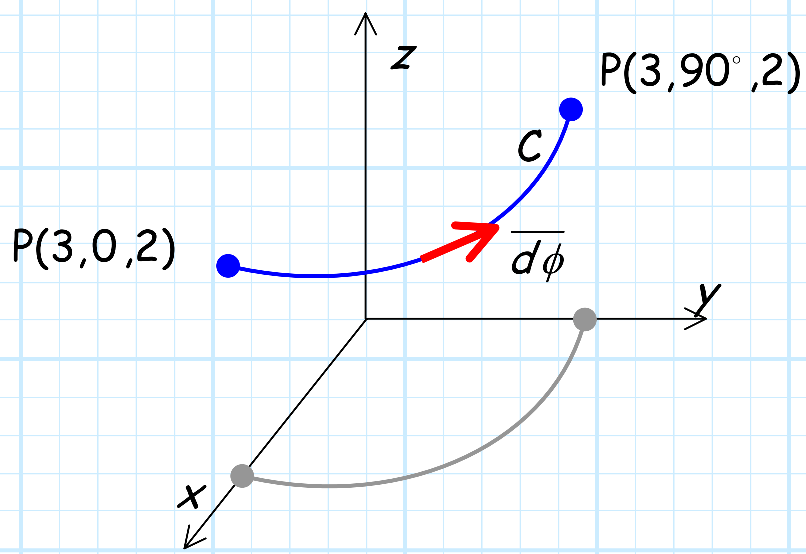


Note: if we move from $\phi = 0$ to $\phi = 360^{\circ}$, a complete **circle** is formed around the z-axis.

Every point along the arc has coordinate values $\rho = 3$ and $z = 2$. As we move along the contour, the **only** coordinate value that changes is ϕ.

Therefore, the **differential** directed distance associated with a change in position from ϕ to $\phi + d\phi$, is:

$$\overline{d\ell} = \overline{d\phi} = \hat{a}_{\phi} \ \ \rho d\phi .$$



Finally, changing coordinate z generates the **third** cylindrical contour—but we **already** did that in Cartesian coordinates! The result is **again** a line segment parallel to the z-axis.

The three cylindrical contours are therefore described as:

1. *Line segment parallel to the z- axis.*

$$\rho = c_\rho \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. *Circular arc parallel to the x-y plane.*

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\phi \;\; \rho d\phi$$

3. *Line segment parallel to the x-y plane.*

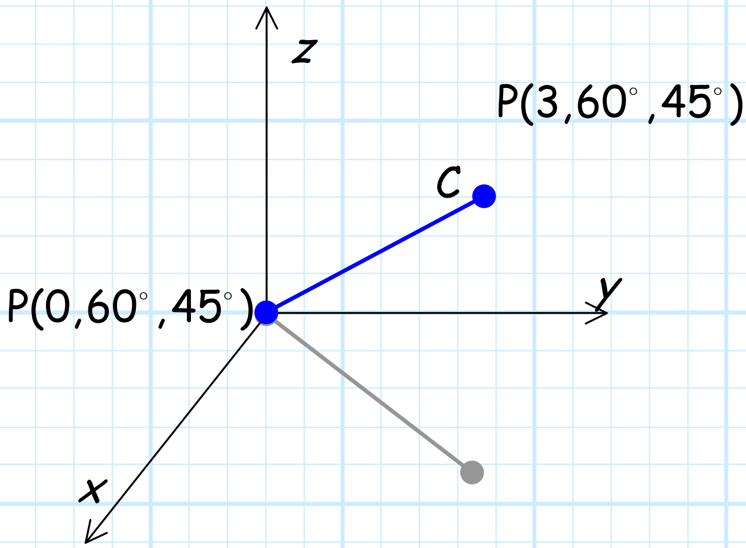
$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\rho \;\; d\rho$$

Spherical Contours

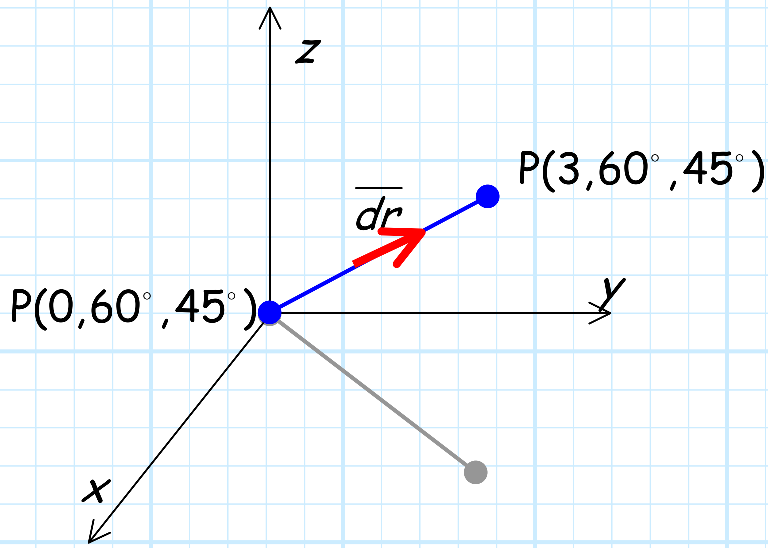
Say we move a point from P(r =0, $\theta = 60^\circ, \phi = 45^\circ$) to P($r$ =3, $\theta = 60^\circ, \phi = 45^\circ$) by changing **only** the coordinate variable r from $r=0$ to r =3. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a contour that is a **line segment**, emerging from the **origin**.



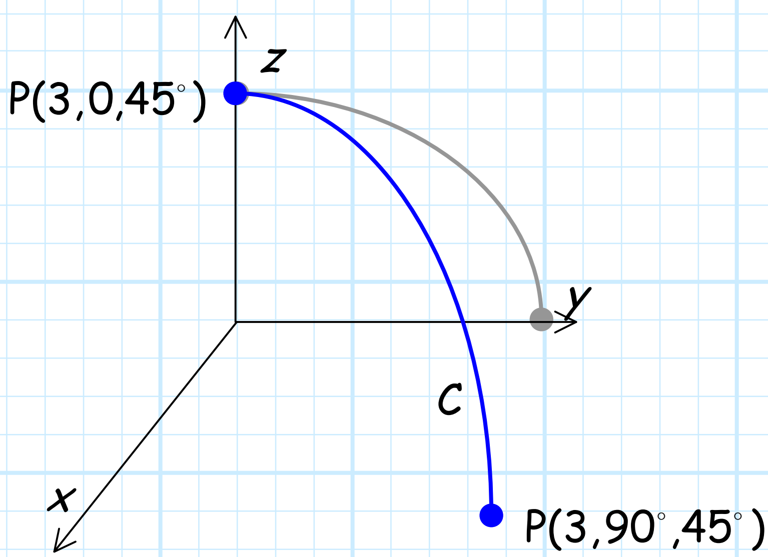
Every point along the line segment has coordinate values $\theta = 60^\circ$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is r.

Therefore, the **differential** directed distance associated with a change in position from r to $r+dr$, is $\overline{d\ell} = \overline{dr} = \hat{a}_r \, dr$.



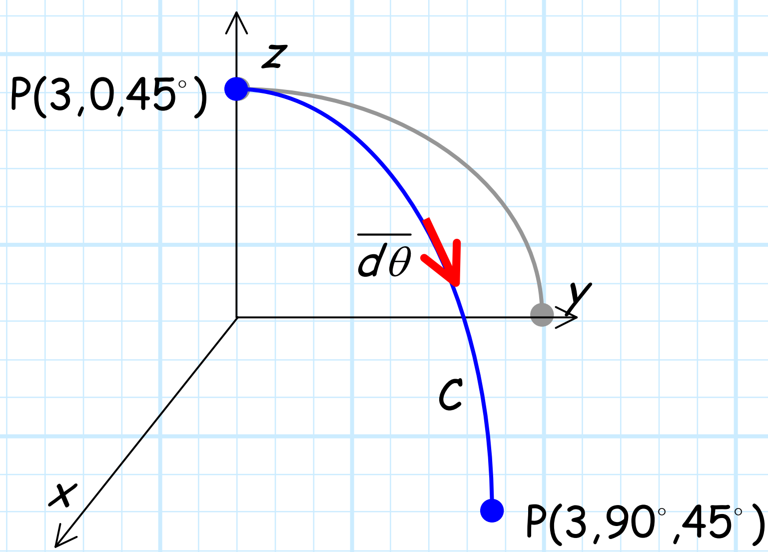
Alternatively, say we move a point from P(r =3, $\theta = 0$,$\phi = 45^\circ$) to P(r =3, $\theta = 90^\circ$,$\phi = 45^\circ$) by changing **only** the coordinate variable θ from $\theta = 0$ to $\theta = 90^\circ$. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a **circular arc**, whose plane includes the *z*-axis.



Every point along the arc has coordinate values $r = 3$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is θ.

Therefore, the **differential** directed distance associated with a change in position from θ to $\theta + d\theta$, is $\overline{d\ell} = \overline{d\theta} = \hat{a}_\theta \, r \, d\theta$.



Finally, we could fix coordinates r and θ and vary coordinate ϕ only—but we **already** did this in cylindrical coordinates! We **again** find that a **circular arc** is generated, an arc that is parallel to the x-y plane.

The three spherical contours are therefore:

1. *A circular arc parallel to the x-y plane.*

$$r = c_r \qquad \theta = c_\theta \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{d\ell} = \hat{a}_\phi \;\; r \sin\theta \, d\phi$$

2. *A circular arc in a plane that includes the z-axis.*

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_\theta \;\; r \, d\theta$$

3. *A line segment directed toward the origin.*

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_\theta \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_r \;\; dr$$