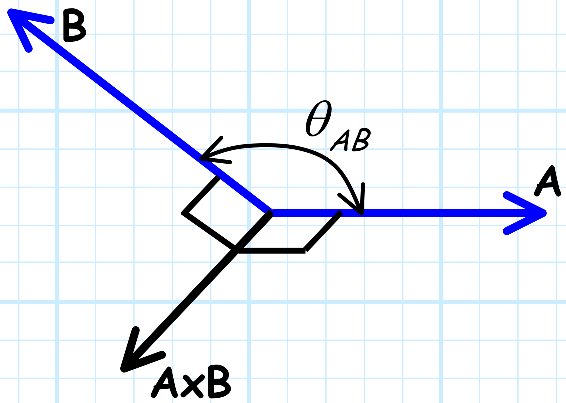
The Cross Product

The **cross product** of two vectors, **A** and **B**, is **denoted** as $\mathbf{A}\times\mathbf{B}$.

The cross product of two vectors is **defined** as:

$$\mathbf{A}\times\mathbf{B} = \hat{a}_n \left|\mathbf{A}\right|\left|\mathbf{B}\right| \sin\theta_{AB}$$

Just as with the dot product, the angle θ_{AB} is the angle between the vectors **A** and **B**. The unit vector $\hat{a}_n$ is **orthogonal** to both **A** and **B** (i.e., $\hat{a}_n \cdot \mathbf{A} = 0$ and $\hat{a}_n \cdot \mathbf{B} = 0$).



$$0 \le \theta_{AB} \le \pi$$

IMPORTANT NOTE: The cross product is an operation involving **two vectors**, and the result is also a **vector**. E.G.,:

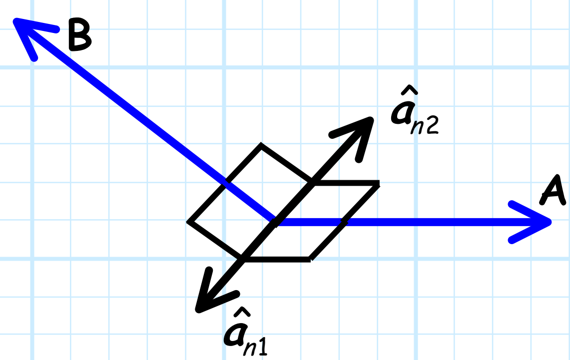
$$\mathbf{A}\times\mathbf{B} = \mathbf{C}$$

The **magnitude** of vector **AxB** is therefore:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta_{AB}$$

Whereas the **direction** of vector **AxB** is described by unit vector $\hat{a}_n$.

Problem! ➡ There are **two** unit vectors that satisfy the equations $\hat{a}_n \cdot \mathbf{A} = 0$ and $\hat{a}_n \cdot \mathbf{B} = 0$!! These two vectors are **anti-parallel**.



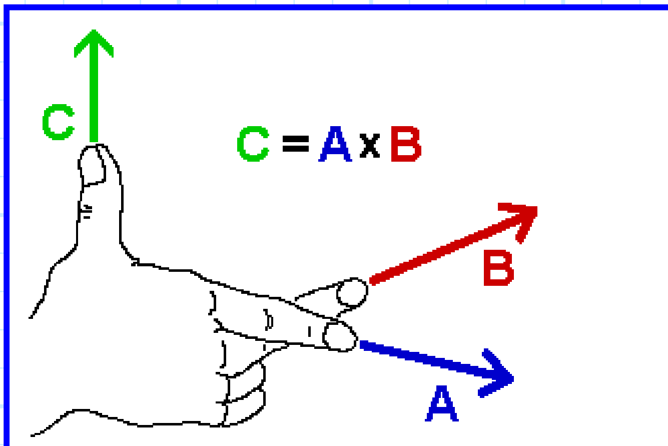
$$\mathbf{A} \cdot \hat{a}_{n1} = \mathbf{A} \cdot \hat{a}_{n2} = 0$$

$$\mathbf{B} \cdot \hat{a}_{n1} = \mathbf{B} \cdot \hat{a}_{n2} = 0$$

$$\hat{a}_{n1} = -\hat{a}_{n2}$$

Q: *Which unit vector is correct?*

A: Use the **right-hand rule** (See figure 2-9)!!

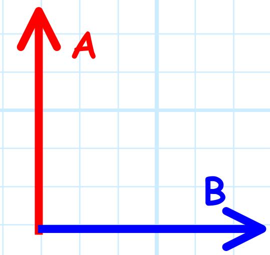


From: www.physics.udel.edu/~watson/phys345/class/1-right-hand-rule.html

Some **fun facts** about the cross product !

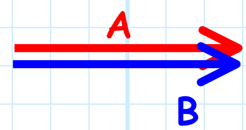
1. If $\theta_{AB} = 90^\circ$ (i.e., **orthogonal**), then:

$$\mathbf{A} \times \mathbf{B} = \hat{a}_n |\mathbf{A}||\mathbf{B}| \sin 90^\circ = \hat{a}_n |\mathbf{A}||\mathbf{B}|$$

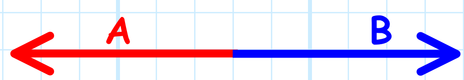


2. If $\theta_{AB} = 0^\circ$ (i.e., **parallel**), then:

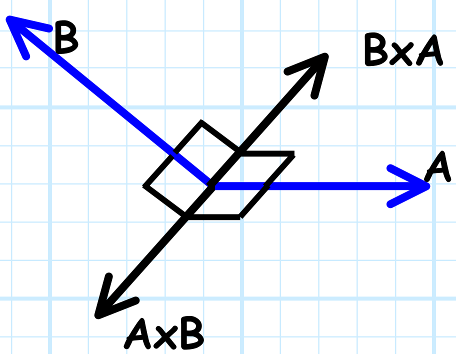
$$\mathbf{A} \times \mathbf{B} = \hat{a}_n |\mathbf{A}||\mathbf{B}| \sin 0^\circ = 0$$

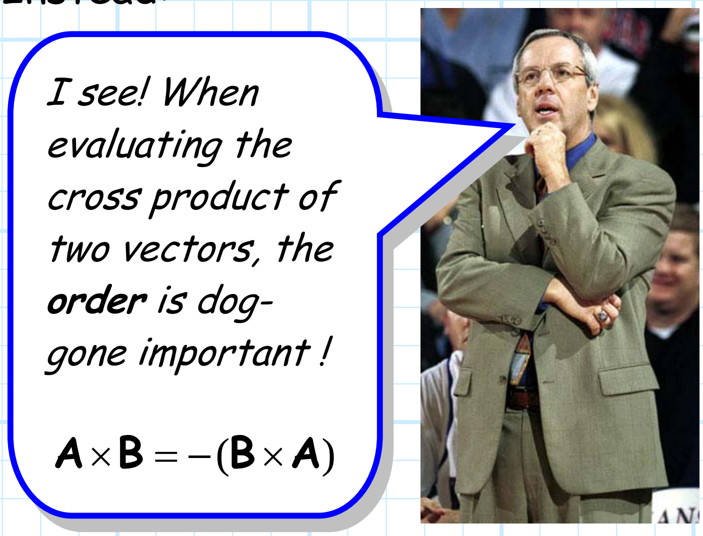


Note that **AxB** =0 also if $\theta_{AB} = 180^\circ$



3. The cross product is **not** commutative ! In other words, $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Instead:





4. The **negative** of the cross product is:

$$-(\mathbf{A} \times \mathbf{B}) = -\mathbf{A} \times \mathbf{B} = \mathbf{A} \times (-\mathbf{B})$$

5. The cross product is also **not** associative:

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

Therefore, **AxBxC** has **ambiguous** meaning !

6. But, the cross product is **distributive**, in that:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

and also,

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$$