The Differential Displacement Vector for Coordinate Systems

Let's determine the differential displacement vectors for each coordinate of the Cartesian, cylindrical and spherical coordinate systems!

**Cartesian**

This is easy!

\[
\frac{d\vec{r}}{dx} = \frac{d\vec{r}}{dx} \, dx = \left[ \left( \frac{d\vec{x}}{dx} \right) \hat{a}_x + \left( \frac{d\vec{y}}{dx} \right) \hat{a}_y + \left( \frac{d\vec{z}}{dx} \right) \hat{a}_z \right] dx = \hat{a}_x \, dx
\]

\[
\frac{d\vec{r}}{dy} = \frac{d\vec{r}}{dy} \, dy = \left[ \left( \frac{d\vec{x}}{dy} \right) \hat{a}_x + \left( \frac{d\vec{y}}{dy} \right) \hat{a}_y + \left( \frac{d\vec{z}}{dy} \right) \hat{a}_z \right] dy = \hat{a}_y \, dy
\]

\[
\frac{d\vec{r}}{dz} = \frac{d\vec{r}}{dz} \, dz = \left[ \left( \frac{d\vec{x}}{dz} \right) \hat{a}_x + \left( \frac{d\vec{y}}{dz} \right) \hat{a}_y + \left( \frac{d\vec{z}}{dz} \right) \hat{a}_z \right] dz = \hat{a}_z \, dz
\]
Cylindrical

Likewise, recall from the last handout that:

\[ d\rho = \hat{\rho} \, d\rho \]

Maria, look! I'm starting to see a trend!

\[
\begin{align*}
\overrightarrow{dx} &= \frac{d\vec{r}}{dx} \, dx = \hat{x} \, dx \\
\overrightarrow{dy} &= \frac{d\vec{r}}{dy} \, dy = \hat{y} \, dy \\
\overrightarrow{dz} &= \frac{d\vec{r}}{dz} \, dz = \hat{z} \, dz \\
\overrightarrow{d\rho} &= \frac{d\vec{r}}{d\rho} \, d\rho = \hat{\rho} \, d\rho
\end{align*}
\]

Q: It seems very apparent that:

\[ \overrightarrow{d\ell} = \hat{\ell} \, d\ell \]

for all coordinates \( \ell \); right?

A: NO!! Do not make this mistake! For example, consider \( d\phi \):

\[
\overrightarrow{d\phi} = \frac{d\vec{r}}{d\phi} \, d\phi
\]

\[
= \left( \frac{dx}{d\phi} \hat{x} + \frac{dy}{d\phi} \hat{y} + \frac{dz}{d\phi} \hat{z} \right) d\phi
\]

\[
= \left( \frac{d\rho \cos \phi}{d\phi} \hat{x} + \frac{d\rho \sin \phi}{d\phi} \hat{y} + \frac{dz}{d\phi} \hat{z} \right) d\phi
\]

\[
= \left( -\rho \sin \phi \hat{x} + \rho \cos \phi \hat{y} \right) \rho \, d\phi = \hat{\phi} \, \rho \, d\phi
\]

Q: No!! \( d\phi = \hat{\phi} \, \rho \, d\phi \) ?!

How did the coordinate \( \rho \) get in there?
The scalar differential value $\rho \, d\phi$ makes sense! The differential displacement vector is a directed distance, thus the units of its magnitude must be distance (e.g., meters, feet). The differential value $d\phi$ has units of radians, but the differential value $\rho \, d\phi$ does have units of distance.

The differential displacement vectors for the cylindrical coordinate system is therefore:

$$
\begin{align*}
\overline{d\rho} &= \frac{d\vec{r}}{d\rho} \, d\rho = \hat{\rho} \, d\rho \\
\overline{d\phi} &= \frac{d\vec{r}}{d\phi} \, d\phi = \hat{\phi} \, \rho \, d\phi \\
\overline{dz} &= \frac{d\vec{r}}{dz} \, dz = \hat{z} \, dz
\end{align*}
$$

Likewise, for the spherical coordinate system, we find that:

$$
\begin{align*}
\overline{dr} &= \frac{d\vec{r}}{dr} \, dr = \hat{r} \, dr \\
\overline{d\theta} &= \frac{d\vec{r}}{d\theta} \, d\theta = \hat{\theta} \, r \, d\theta \\
\overline{d\phi} &= \frac{d\vec{r}}{d\phi} \, d\phi = \hat{\phi} \, r \sin \theta \, d\phi
\end{align*}
$$