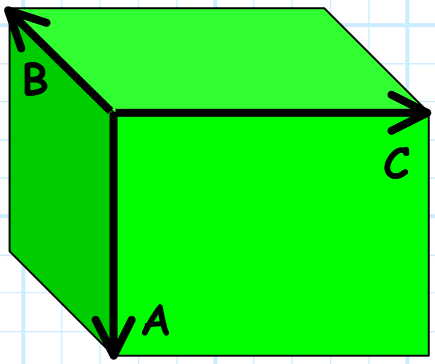
The Differential Volume Element

Consider a **rectangular cube**, whose **three** sides can be defined by **three** directed distances, say **A**, **B**, and **C**.



It is evident that the lengths of each side of the rectangular cube are $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{C}|$, such that the **volume** of this rectangular cube can be expressed as:

$$V = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now what happens if we take the **triple product** of these three vectors:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\cdot\hat{a}_n|\mathbf{B}||\mathbf{C}|\sin\theta_{BC}$$

However, we note that $\sin\theta_{BC} = \sin 90^\circ = 1.0$, and that $\hat{a}_n = \hat{a}_A$ (i.e., vector $\mathbf{B}\times\mathbf{C}$ points in the same direction as vector **A**!).

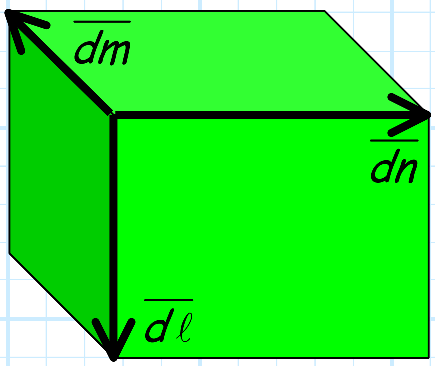
Using the fact that $\mathbf{A} = |\mathbf{A}|\,\hat{a}_A$, we then find the result:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} &= \mathbf{A}\cdot\hat{a}_n\,|\mathbf{B}||\mathbf{C}|\sin\theta_{BC}\\ &= \mathbf{A}\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}|\\ &= |\mathbf{A}|\,\hat{a}_A\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}|\\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\end{aligned}$$

Look what this means, the **volume** of a cube can be expressed in terms of the **triple product**!

$$V = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now a rectangular volume formed by three orthogonal **line vectors** (e.g., $\overline{dx}$, $\overline{dy}$, $\overline{dz}$ or $\overline{d\rho}$, $\overline{d\phi}$, $\overline{dz}$).



The result is a differential volume, given as:

$$dv = \overline{d\ell}\cdot\overline{dm}\times\overline{dn}$$

For example, for the **Cartesian** coordinate system:

$$\begin{aligned} dv &= \overline{dx} \cdot \overline{dy} \times \overline{dz} \\ &= dx\ dy\ dz \end{aligned}$$

and for the **cylindrical** coordinate system:

$$\begin{aligned} dv &= \overline{d\rho} \cdot \overline{d\phi} \times \overline{dz} \\ &= \rho\ d\rho\ d\phi\ dz \end{aligned}$$

and also for the **spherical** coordinate system:

$$\begin{aligned} dv &= \overline{dr} \cdot \overline{d\theta} \times \overline{d\phi} \\ &= r^2 \sin\theta\ dr\ d\phi\ d\theta \end{aligned}$$