The Differential Form of the Lorentz Force Law

Say that instead of a single charged particle, we have a small volume \( dv \) of charge, with volume charge density \( \rho_v(\vec{r}) \).

Therefore:

\[
dQ = \rho_v(\vec{r}) \ dv
\]

If this volume of charge is also moving with a velocity \( \vec{u} \), then the force \( d\vec{F}(\vec{r}) \) on charge \( dQ \) will be:

\[
d\vec{F}(\vec{r}) = dQ(\vec{E}(\vec{r}) + \vec{u} \times \vec{B}(\vec{r})))
\]

\[
= \rho(\vec{r}) \vec{E}(\vec{r}) \ dv + \rho(\vec{r}) \vec{u} \times \vec{B}(\vec{r}) \ dv
\]

\[
= \rho(\vec{r}) \vec{E}(\vec{r}) \ dv + \vec{J}(\vec{r}) \times \vec{B}(\vec{r}) \ dv
\]

where we recall that \( \vec{J}(\vec{r}) = \rho_v(\vec{r}) \vec{u} \).

Therefore we can state that:

\[
d\vec{F}_e(\vec{r}) = \rho_v(\vec{r}) \vec{E}(\vec{r}) \ dv
\]

\[
d\vec{F}_m(\vec{r}) = \vec{J}(\vec{r}) \times \vec{B}(\vec{r}) \ dv
\]

Look what this means!
It means that not only does an electric field apply a force to a single charge $Q$, but it also applies a force on a whole collection of charges, described by $\rho_v(\vec{r})$.

Likewise, the magnetic flux density not only applies a force on a moving charge particle, it also applies a force to any current distribution described by $\vec{J}(\vec{r})$.

To determine the total force on some volume $V$ enclosing charges and currents, we must integrate over the entire volume:

\[
\mathbf{F} = \iiint_V d\mathbf{F}(\vec{r}) dV
\]

\[
= \iiint_V (\rho(\vec{r}) \mathbf{E}(\vec{r}) + \mathbf{J}(\vec{r}) \times \mathbf{B}(\vec{r})) dV
\]

The Lorentz force equation tells us how fields $\mathbf{E}(\vec{r})$ and $\mathbf{B}(\vec{r})$ affect charges $\rho_v(\vec{r})$ and currents $\mathbf{J}(\vec{r})$.

But remember, charges $\rho_v(\vec{r})$ and currents $\mathbf{J}(\vec{r})$ likewise create fields $\mathbf{E}(\vec{r})$ and $\mathbf{B}(\vec{r})$!

Q: *How can we determine what fields $\mathbf{E}(\vec{r})$ and $\mathbf{B}(\vec{r})$ are created by charges $\rho_v(\vec{r})$ and currents $\mathbf{J}(\vec{r})$?*

A: Ask Jimmy Maxwell!