The Differential Form of the Lorentz Force Law

Say that instead of a single charged particle, we have a small **volume** dv of charge, with volume **charge density** $\rho_v(\bar{r})$. Therefore:

$$dQ = \rho_v(\bar{r})\ dv$$

If this volume of charge is also moving with a velocity $\mathbf{u}$, then the force $\mathbf{dF}(\bar{r})$ on charge dQ will be:

$$\begin{aligned}\mathbf{dF}(\bar{r}) &= dQ\left(\mathbf{E}(\bar{r}) + \mathbf{u}\times\mathbf{B}(\bar{r})\right)\\ &= \rho(\bar{r})\,\mathbf{E}(\bar{r})\,dv + \rho(\bar{r})\,\mathbf{u}\times\mathbf{B}(\bar{r})\,dv\\ &= \rho(\bar{r})\,\mathbf{E}(\bar{r})\,dv + \mathbf{J}(\bar{r})\times\mathbf{B}(\bar{r})\,dv\end{aligned}$$

where we recall that $\mathbf{J}(\bar{r}) = \rho_v(\bar{r})\,\mathbf{u}$.

Therefore we can state that:

$$\mathbf{dF_e}(\bar{r}) = \rho_v(\bar{r})\,\mathbf{E}(\bar{r})\,dv$$

$$\mathbf{dF}_m(\bar{r}) = \mathbf{J}(\bar{r})\times\mathbf{B}(\bar{r})\,dv$$

Look what this means!

It means that not only does an electric field apply a force to a single charge Q, but it also applies a force on a whole **collection** of charges, described by $\rho_v(\bar{r})$.

Likewise, the magnetic flux density not only applies a force on a moving charge particle, it also applies a force to **any** current distribution described by $\mathbf{J}(\bar{r})$.

To determine the **total** force on some **volume** V enclosing charges and currents, we must **integrate** over the entire volume:

$$\begin{aligned}\mathbf{F} &= \iiint_V d\mathbf{F}(\bar{r})\,dv \\ &= \iiint_V \left(\rho(\bar{r})\,\mathbf{E}(\bar{r}) + \mathbf{J}(\bar{r})\times\mathbf{B}(\bar{r})\right) dv\end{aligned}$$

The Lorentz force equation tells us how fields $\mathbf{E}(\bar{r})$ and $\mathbf{B}(\bar{r})$ **affect** charges $\rho_v(\bar{r})$ and currents $\mathbf{J}(\bar{r})$.

But remember, charges $\rho_v(\bar{r})$ and currents $\mathbf{J}(\bar{r})$ likewise **create** fields $\mathbf{E}(\bar{r})$ and $\mathbf{B}(\bar{r})$!

Q: *How can we determine what fields* $\mathbf{E}(\bar{r})$ and $\mathbf{B}(\bar{r})$ *are **created** by charges* $\rho_v(\bar{r})$ *and currents* $\mathbf{J}(\bar{r})$*?*

A: Ask Jimmy **Maxwell**!