The Divergence Theorem

Recall we studied volume integrals of the form:

\[ \iiint_{V} g(\vec{r}) \, dv \]

It turns out that any and every scalar field can be written as the divergence of some vector field, i.e.:

\[ g(\vec{r}) = \nabla \cdot \mathbf{A}(\vec{r}) \]

Therefore we can equivalently write any volume integral as:

\[ \iiint_{V} \nabla \cdot \mathbf{A}(\vec{r}) \, dv \]

The divergence theorem states that this integral is equal to:

\[ \iiint_{V} \nabla \cdot \mathbf{A}(\vec{r}) \, dv = \iint_{S} \mathbf{A}(\vec{r}) \cdot d\mathbf{s} \]

where \( S \) is the closed surface that completely surrounds volume \( V \), and vector \( d\mathbf{s} \) points outward from the closed surface. For example, if volume \( V \) is a sphere, then \( S \) is the surface of that sphere.

The divergence theorem states that the volume integral of a scalar field can be likewise evaluated as a surface integral of a vector field!
What the divergence theorem indicates is that the total “divergence” of a vector field through the surface of any volume is equal to the sum (i.e., integration) of the divergence at all points within the volume.

In other words, if the vector field is diverging from some point in the volume, it must simultaneously be converging to another adjacent point within the volume—the net effect is therefore zero!

Thus, the only values that make any difference in the volume integral are the divergence or convergence of the vector field across the surface surrounding the volume—vectors that will be converging or diverging to adjacent points outside the volume (across the surface) from points inside the volume. Since these points just outside the volume are not included in the integration, their net effect is non-zero!