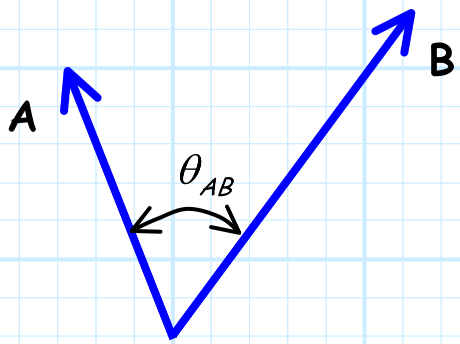
The Dot Product

The **dot product** of two vectors, A and B, is **denoted** as $\mathbf{A} \cdot \mathbf{B}$.

The dot product of two vectors is **defined** as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta_{AB}$$

where the angle θ_{AB} is the angle formed **between** the vectors **A** and **B**.



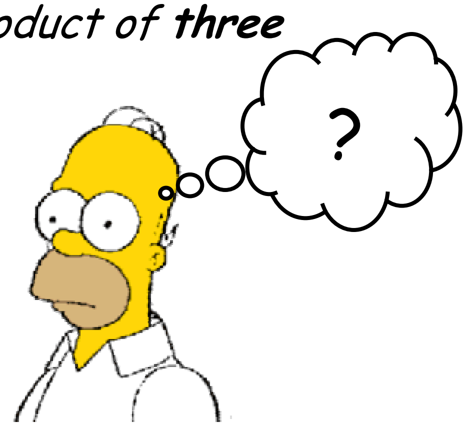
$$0 \leq \theta_{AB} \leq \pi$$

IMPORTANT NOTE: The dot product is an operation involving **two vectors**, but the result is a **scalar** !! E.G.,:

$$\mathbf{A} \cdot \mathbf{B} = c$$

The dot product is also called the **scalar product** of two vectors.

Q1: *So, what would the dot product of **three** vectors be? I.E.,:*



$$\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = ??$$

A:

Note also that the dot product is **commutative**, i.e.,:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

Some **more** fun facts about the dot product include:

1. The dot product of a vector **with itself** is equal to the **magnitude** of the vector **squared**.

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos 0^{\circ} = |\mathbf{A}|^2$$

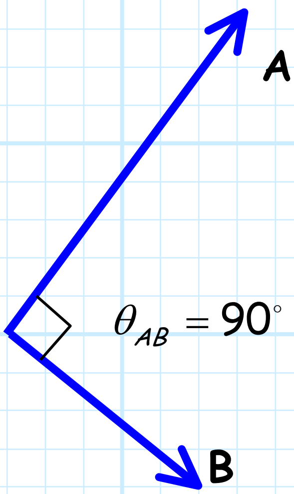
Therefore:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

2. If $\mathbf{A} \bullet \mathbf{B} = 0$ (and $|\mathbf{A}| \neq 0, |\mathbf{B}| \neq 0$), then it must be true that:

$$\cos\theta_{AB} = 0 \quad \Rightarrow \quad \theta_{AB} = 90^\circ$$

Thus, if $\mathbf{A} \bullet \mathbf{B} = 0$, the two vectors are **orthogonal** (perpendicular).

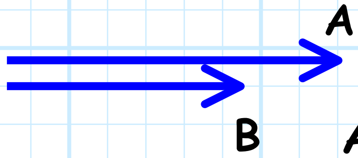


$$\mathbf{A} \bullet \mathbf{B} = 0$$

3. If $\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|$, then it must be true that:

$$\cos\theta_{AB} = 1 \quad \Rightarrow \quad \theta_{AB} = 0$$

Thus, vectors **A** and **B** must have the **same direction**. They are said to be **collinear** (parallel).

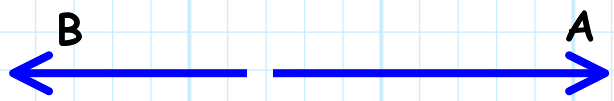


$$\begin{aligned} \mathbf{A} \bullet \mathbf{B} &= |\mathbf{A}||\mathbf{B}|\cos 0 \\ &= |\mathbf{A}||\mathbf{B}| \end{aligned}$$

4. If $\mathbf{A} \bullet \mathbf{B} = -|\mathbf{A}||\mathbf{B}|$, then it must be true that:

$$\cos\theta_{AB} = -1 \quad \Rightarrow \quad \theta_{AB} = 180^{\circ}$$

Thus, vectors **A** and **B** point in **opposite** directions; they are said to be **anti-parallel**.



$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= |\mathbf{A}||\mathbf{B}| \cos 180^{\circ} \\ &= -|\mathbf{A}||\mathbf{B}|\end{aligned}$$

5. The dot product is **distributive** with addition, such that:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

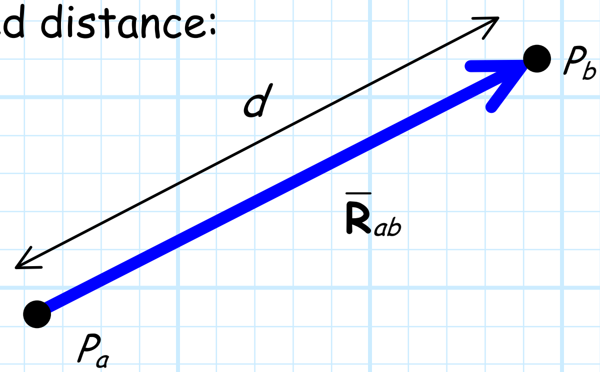
For example, we can write:

$$\begin{aligned}(\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B}) \cdot \mathbf{A} + (\mathbf{A}+\mathbf{B}) \cdot \mathbf{C} && \text{(distributive)} \\ &= \mathbf{A} \cdot (\mathbf{A}+\mathbf{B}) + \mathbf{C} \cdot (\mathbf{A}+\mathbf{B}) && \text{(commutative)} \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B} && \text{(distributive)} \\ &= |\mathbf{A}|^2 + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B}\end{aligned}$$

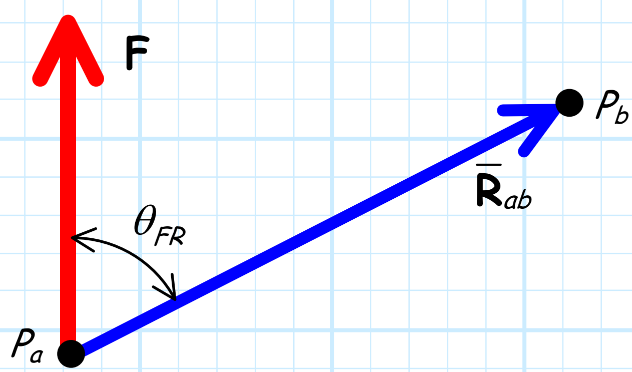
One application of the dot product is the determination of **work**. Say an object moves a distance d, directly from point P_a to point P_b , by applying a constant force **F**.

Q: *How much work has been done?*

First, we can specify the direct path from point P_a to point P_b with a directed distance:



The work done is simply the **dot product** of the applied force vector and the directed distance!



A:

$$\begin{aligned} W &= \mathbf{F} \cdot \overline{\mathbf{R}}_{ab} \\ &= |\mathbf{F}| \, |\overline{\mathbf{R}}_{ab}| \cos\theta_{FR} \\ &= d \, |\mathbf{F}| \cos\theta_{FR} \end{aligned}$$

The value $|\mathbf{F}|\cos\theta_{FR}$ is said to be the **scalar component** of force **F** in the **direction** of directed distance $\overline{\mathbf{R}}_{ab}$