The Dot Product

The dot product of two vectors, A and B, is denoted as $A \cdot B$.

The dot product of two vectors is defined as:

$$A \cdot B = |A||B| \cos \theta_{AB}$$

where the angle $\theta_{AB}$ is the angle formed between the vectors A and B.

**IMPORTANT NOTE:** The dot product is an operation involving two vectors, but the result is a scalar!! E.G.,:

$$A \cdot B = c$$

The dot product is also called the scalar product of two vectors.
Q1: So, what would the dot product of three vectors be? I.E.,:

\[ \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = ?? \]

A:

Note also that the dot product is \textit{commutative}, i.e.,:

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \]

Some more fun facts about the dot product include:

1. The dot product of a vector \textit{with itself} is equal to the magnitude of the vector squared.

\[ \mathbf{A} \cdot \mathbf{A} = |\mathbf{A}| |\mathbf{A}| \cos 0^\circ = |\mathbf{A}|^2 \]

Therefore:

\[ |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \]
2. If \( \mathbf{A} \cdot \mathbf{B} = 0 \) (and \( |\mathbf{A}| \neq 0, |\mathbf{B}| \neq 0 \)), then it must be true that:

\[
\cos \theta_{\mathbf{A}\mathbf{B}} = 0 \quad \Rightarrow \quad \theta_{\mathbf{A}\mathbf{B}} = 90^\circ
\]

Thus, if \( \mathbf{A} \cdot \mathbf{B} = 0 \), the two vectors are **orthogonal** (perpendicular).

![Diagram of two orthogonal vectors](image)

3. If \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \), then it must be true that:

\[
\cos \theta_{\mathbf{A}\mathbf{B}} = 1 \quad \Rightarrow \quad \theta_{\mathbf{A}\mathbf{B}} = 0
\]

Thus, vectors \( \mathbf{A} \) and \( \mathbf{B} \) must have the same direction. They are said to be **collinear** (parallel).

![Diagram of collinear vectors](image)
4. If \( \mathbf{A} \cdot \mathbf{B} = -|\mathbf{A}| |\mathbf{B}| \), then it must be true that:

\[
\cos \theta_{AB} = -1 \implies \theta_{AB} = 180^\circ
\]

Thus, vectors \( \mathbf{A} \) and \( \mathbf{B} \) point in opposite directions; they are said to be anti-parallel.

\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos 180^\circ = -|\mathbf{A}| |\mathbf{B}|
\]

5. The dot product is distributive with addition, such that:

\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}
\]

For example, we can write:

\[
(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) \cdot \mathbf{A} + (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} \quad \text{(distributive)}
= \mathbf{A} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{C} \cdot (\mathbf{A} + \mathbf{B}) \quad \text{(commutative)}
= \mathbf{A} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B} \quad \text{(distributive)}
= |\mathbf{A}|^2 + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B}
\]
One application of the dot product is the determination of **work**. Say an object moves a distance \( d \), directly from point \( P_a \) to point \( P_b \), by applying a constant force \( F \).

**Q:** *How much work has been done?*

First, we can specify the direct path from point \( P_a \) to point \( P_b \) with a directed distance:

The work done is simply the **dot product** of the applied force vector and the directed distance!

\[
\mathbf{W} = \mathbf{F} \cdot \mathbf{R}_{ab}
\]

\[
= |\mathbf{F}| |\mathbf{R}_{ab}| \cos \theta_{FR}
\]

\[
= d |\mathbf{F}| \cos \theta_{FR}
\]

The value \( |\mathbf{F}| \cos \theta_{FR} \) is said to be the **scalar component** of force \( \mathbf{F} \) in the **direction** of directed distance \( \mathbf{R}_{ab} \).