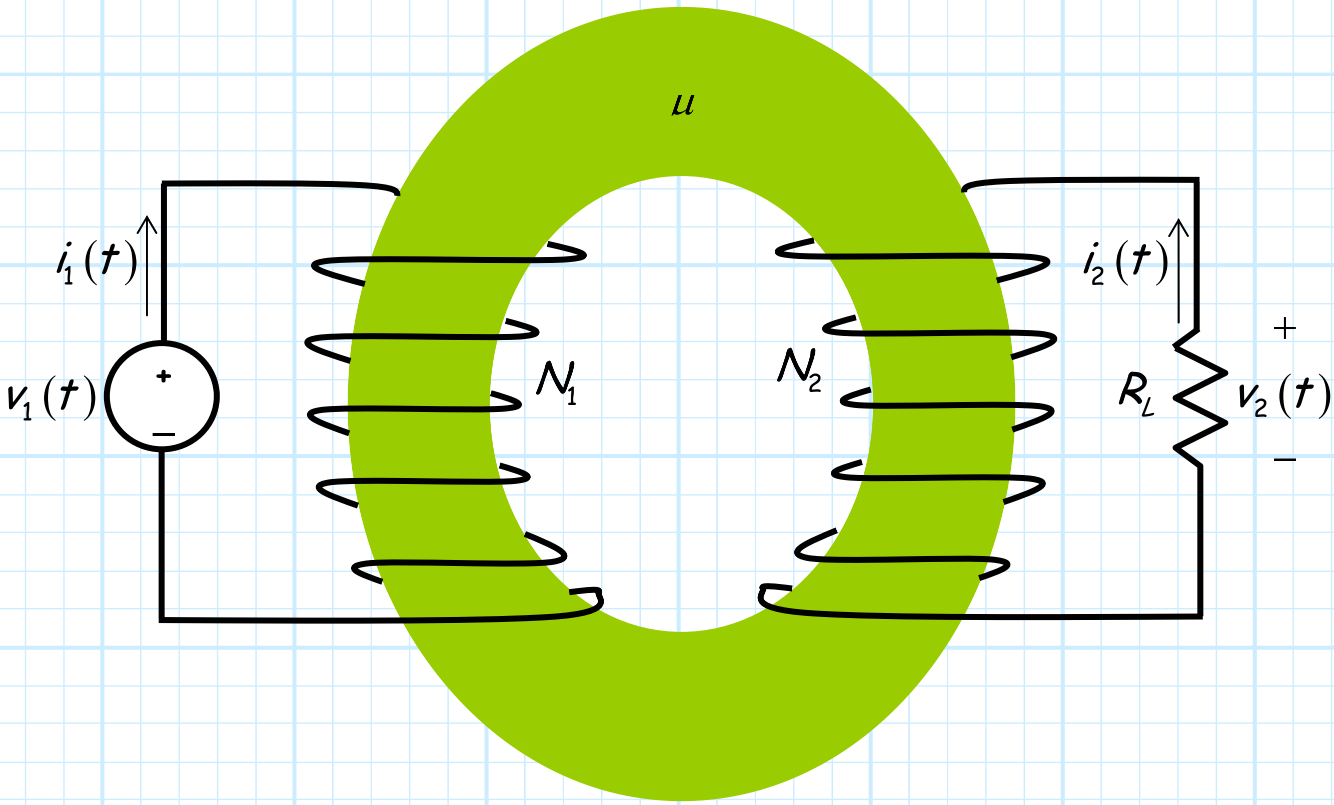
The Ideal Transformer

Consider the structure:



* The "doughnut" is a ring made of **magnetic material** with **very** large relative permeability (i.e., $\mu_r >> 1$).

* On one side of the ring is a coil of wire with N_1 turns. This could of wire forms a **solenoid**!

* On the other side of the ring is **another** solenoid, consisting of a coil of N_2 turns.

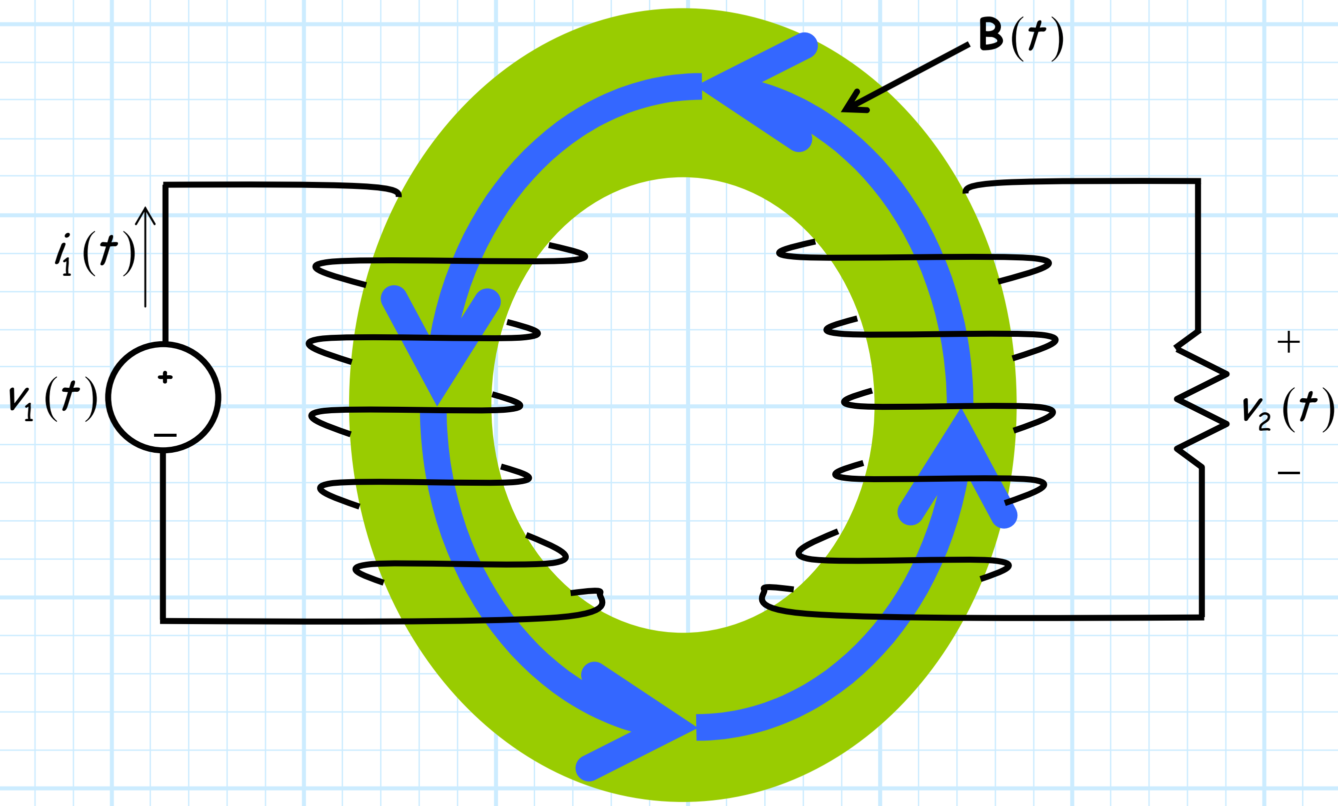
This structure is an **ideal transformer** !

* The solenoid on the left is the **primary loop**, where the one on the right is called the **secondary loop**.

The current $i_1(t)$ in the primary generates a **magnetic flux density** $\mathbf{B}(\bar{r},t)$. Recall for a solenoid, this flux density is approximately **constant** across the solenoid cross-section (i.e., with respect to $\bar{r}$). Therefore, we find that the magnetic flux density within the solenoid can be written as:

$$\mathbf{B}(\bar{r},t) = \mathbf{B}(t)$$

It turns out, since the permeability of the ring is **very large**, then this flux density will be **contained** almost entirely **within** the magnetic ring.



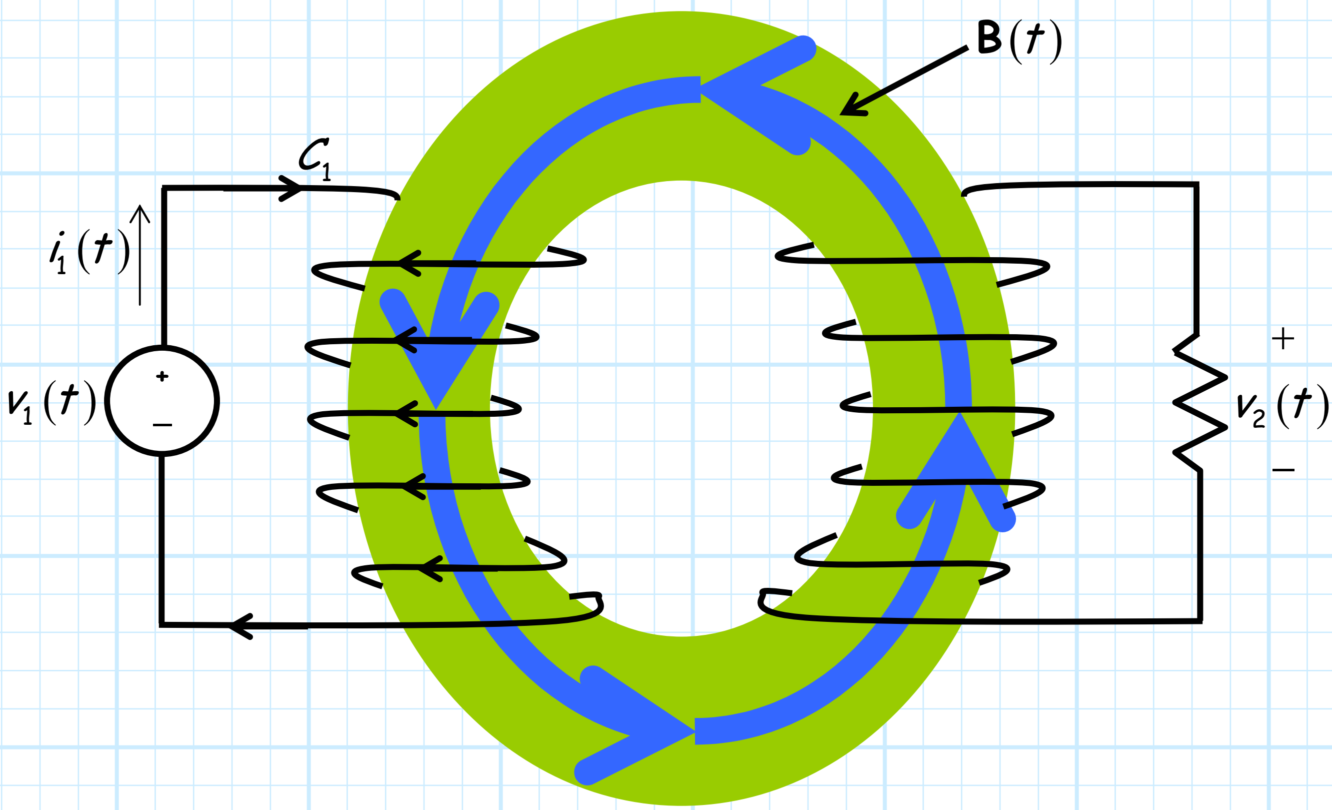
Therefore, we find that the magnetic flux density in the **secondary** solenoid is **equal** to that produced in the **primary**!

Q: *Does this mean also that* $v_1(t) = v_2(t)$*?*

A: Let's apply **Faraday's Law** and find out!

Applying Faraday's Law to the **primary** loop, defined as **contour** C_1, we get:

$$\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint_{S_1} \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$



Q: *But, **contour** C_1 follows the wire of the **solenoid**. What the heck then is **surface** S_1??*

A: S_1 is the surface of a **spiral**!

We can approximate the **surface area** of a spiral by first considering the surface area formed by a **single loop** of wire, denoted S_0. The surface area of a spiral of N **turns** is therefore approximately N S_0. Thus, we say:

$$\iint_{S_1} \mathbf{B}(t) \cdot \overline{ds} = N_1 \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds}$$

Likewise, we find that by integrating around **contour** C_1:

$$-\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v_1(t)$$

Faraday's Law therefore becomes:

$$\begin{aligned} v_1(t) &= N_1 \frac{\partial}{\partial t} \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds} \\ &= N_1 \frac{\partial \Phi(t)}{\partial t} \end{aligned}$$

where $\Phi(t)$ is the total **magnetic flux** flowing through the solenoid:

$$\Phi(t) = \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds}$$

Remember, this **same** magnetic flux is flowing through the **secondary** solenoid as well. Faraday's Law for this solenoid is:

$$\oint_{C_2} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint_{S_2} \mathbf{B}(\bar{r}, t) \cdot \overline{ds}$$

where we similarly find that:

$$-\oint_{C_2} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v_2(t)$$

and:

$$\begin{aligned} \frac{\partial}{\partial t} \iint_{S_2} \mathbf{B}(\bar{r}, t) \cdot \overline{ds} &= N_2 \frac{\partial}{\partial t} \iint_{S_0} \mathbf{B}(\bar{r}, t) \cdot \overline{ds} \\ &= N_2 \frac{\partial \Phi(t)}{\partial t} \end{aligned}$$

therefore we find that :

$$v_2(t) = N_2 \frac{\partial \Phi(t)}{\partial t}$$

Combining this with our expression for the primary, we get:

$$\frac{\partial \Phi(t)}{\partial t} = \frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$

As a result, we find that the voltage $v_2(t)$ across the **load resistor** R_L is related to the voltage **source** $v_1(t)$ as:

$$v_2(t) = \frac{N_2}{N_1} v_1(t)$$

Note that by changing the number of the **ratio** of windings N in each solenoid, a transformer can be constructed such that the output voltage $v_2(t)$ is either much **greater** than the input voltage $v_1(t)$(i.e., $N_2/N_1 >> 1$), or much **less** than the input voltage (i.e., $N_2/N_1 << 1$).

We call the first case a **step-up** transformer, and the later case a **step-down** transformer.

Q: *How are the currents $i_1(t)$ and $i_2(t)$ related ??*

A: Energy must be **conserved**!

Since a transformer is a **passive** device, it cannot **create** energy. We can state therefore that the power **absorbed** by the resistor must be equal to the power **delivered** by the voltage source.

In other words:

$$Power = v_1(t)\, i_1(t) = -v_2(t)\, i_2(t)$$

The minus sign in the above expression comes from the definition of $i_2(t)$, which is pointing **into** the transformer (as opposed to pointing into the resistor).

Rearranging the above expression, we find:

$$\begin{aligned} i_2(t) &= -\frac{v_1(t)}{v_2(t)}\, i_1(t) \\ &= -\frac{N_1}{N_2}\, i_1(t) \end{aligned}$$

Note that for a **step-up** transformer, the output current $i_2(t)$ is actually **less** than that of $i_1(t)$, whereas for the step-down transformer the opposite is true.

Thus, if the **voltage is increased**, the **current is decreased** proportionally—energy is conserved!

Finally, we note that the primary of the transformer has the apparent **resistance** of:

$$R_1 \doteq \frac{v_1}{i_1} = \frac{v_1}{v_2}\frac{v_2}{i_2}\frac{i_2}{i_1} = \frac{N_1}{N_2}(-R_L)\frac{-N_1}{N_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Thus, we find that for a **step-up** transformer, the primary resistance is much **greater** than that of the load resistance on the secondary. Conversely, a **step-down** transformer will exhibit a primary resistance R_1 that is much **smaller** than that of the load.

One more **important** note! We applied conservation of energy to this problem because a transformer is a passive device. Unlike an active device (e.g., current or voltage source) it cannot **add** energy to the system .

However, passive devices can certainly **extract** energy from the system!

Q: *How can they do this?*

A: They can convert electromagnetic energy to **heat** !

If the "doughnut" is lossy (i.e., **conductive**), electric **currents** $\mathbf{J}(\bar{r})$ can be **induced** in the magnetic material. The result are **ohmic losses**, which is power delivered to some volume V (e.g., the doughnut) and then converted to **heat**. This loss can be determined from **Joule's Law**:

$$P_{loss} = \iiint_V \sigma \left|\mathbf{E}(\bar{r})\right|^2 dv \qquad [W]$$

In this case, the transformer is **non-ideal**, and the expressions derived in this handout are only **approximate**.