The Integral Form of Electrostatics

We know from the **static** form of Maxwell's equations that the vector field $\nabla x \mathbf{E}(\bar{r})$ is zero at every point $\bar{r}$ in space (i.e., $\nabla x \mathbf{E}(\bar{r})=0$). Therefore, **any** surface integral involving the vector field $\nabla x \mathbf{E}(\bar{r})$ will likewise be zero:

$$\iint_S \nabla x \mathbf{E}(\bar{r}) \cdot \overline{ds} = 0$$

But, using **Stokes' Theorem**, we can also write:

$$\iint_S \nabla x \mathbf{E}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

Therefore, the equation:

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

is the **integral form** of the equation:

$$\nabla x \mathbf{E}(\bar{r}) = 0$$

Of course, both equations just indicate that the **static electric field** $\mathbf{E}(\bar{r})$ is a **conservative** field!

Likewise, we can take a volume integral over both sides of the electrostatic equation $\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})/\varepsilon_0$:

$$\iiint_V \nabla \cdot \mathbf{E}(\bar{\mathbf{r}})\, dv = \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{\mathbf{r}})\, dv$$

But wait! The left side can be rewritten using the **Divergence Theorem**:

$$\iiint_V \nabla \cdot \mathbf{E}(\bar{\mathbf{r}})\, dv = \oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds}$$

And, we know that the volume integral of the charge density is equal to the **charge enclosed** in volume V:

$$\iiint_V \rho_v(\bar{\mathbf{r}})\, dv = Q_{enc}$$

Therefore, we can write an equation known as **Gauss's Law**:

$$\oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$ **Gauss's Law**

This is the **integral form** of the equation $\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})/\varepsilon_0$.

What **Gauss's Law** says is that we can determine the total amount of **charge enclosed** within some **volume** V by simply integrating the electric field on the surface S surrounding volume V.

Summarizing, the **integral form** of the electrostatic equations are:

$$\oint_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} = 0 \qquad \oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds} = \frac{Q}{\varepsilon_0}$$

Note that these equations do **not** amend or extend what we already know about the static electric field, but are simply an **alternative** way of expressing the **point** form of the electrostatic equations:

$$\nabla \times \mathbf{E}(\bar{\mathbf{r}}) = 0$$

$$\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

We **sometimes** use the **point** form of the electrostatic equations, and we sometimes use the **integral** form—it all depends on which form is more applicable to the problem we are attempting to solve!