The Integral Form of Electrostatics

We know from the static form of Maxwell’s equations that the vector field $\nabla \times \mathbf{E}(\mathbf{r})$ is zero at every point $\mathbf{r}$ in space (i.e., $\nabla \times \mathbf{E}(\mathbf{r}) = 0$). Therefore, any surface integral involving the vector field $\nabla \times \mathbf{E}(\mathbf{r})$ will likewise be zero:

$$\int \int_{S} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = 0$$

But, using Stokes' Theorem, we can also write:

$$\int \int_{S} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = \oint_{C} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = 0$$

Therefore, the equation:

$$\oint_{C} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = 0$$

is the integral form of the equation:

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

Of course, both equations just indicate that the static electric field $\mathbf{E}(\mathbf{r})$ is a conservative field!
Likewise, we can take a volume integral over both sides of the electrostatic equation \( \nabla \cdot \mathbf{E}(\mathbf{r}) = \rho_v(\mathbf{r})/\varepsilon_0 \):

\[
\iiint_{V} \nabla \cdot \mathbf{E}(\mathbf{r}) \, d\mathbf{r} = \frac{1}{\varepsilon_0} \iiint_{V} \rho_v(\mathbf{r}) \, d\mathbf{r}
\]

But wait! The left side can be rewritten using the **Divergence Theorem**:

\[
\iiint_{V} \nabla \cdot \mathbf{E}(\mathbf{r}) \, d\mathbf{r} = \oiint_{S} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s}
\]

And, we know that the volume integral of the charge density is equal to the **charge enclosed** in volume \( V \):

\[
\iiint_{V} \rho_v(\mathbf{r}) \, d\mathbf{r} = Q_{\text{enc}}
\]

Therefore, we can write an equation known as **Gauss's Law**:

\[
\oiint_{S} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

This is the **integral form** of the equation \( \nabla \cdot \mathbf{E}(\mathbf{r}) = \rho_v(\mathbf{r})/\varepsilon_0 \).

What **Gauss's Law** says is that we can determine the total amount of charge enclosed within some volume \( V \) by simply integrating the electric field on the surface \( S \) surrounding volume \( V \).
Summarizing, the **integral form** of the electrostatic equations are:

\[
\oint_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = 0 \quad \oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}
\]

Note that these equations do **not** amend or extend what we already know about the static electric field, but are simply an **alternative** way of expressing the **point** form of the electrostatic equations:

\[
\nabla \times \mathbf{E}(\mathbf{r}) = 0 \\
\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_v(\mathbf{r})}{\varepsilon_0}
\]

We sometimes use the **point** form of the electrostatic equations, and we sometimes use the **integral** form—it all depends on which form is more applicable to the problem we are attempting to solve!