The Laplacian

Another differential operator used in electromagnetics is the **Laplacian** operator. There is both a **scalar** Laplacian operator, and a **vector** Laplacian operator. Both operations, however, are expressed in terms of derivative operations that we have **already** studied !

The Scalar Laplacian

The scalar Laplacian is simply the **divergence** of the **gradient** of a scalar field:

$$\nabla \cdot \nabla g(\bar{\mathbf{r}})$$

The scalar Laplacian therefore both **operates** on a scalar field and **results** in a scalar field.

Often, the Laplacian is denoted as "∇^2", i.e.:

$$\nabla^2 g(\bar{\mathbf{r}}) \doteq \nabla \cdot \nabla g(\bar{\mathbf{r}})$$

From the expressions of divergence and gradient, we find that the scalar Laplacian is expressed in **Cartesian** coordinates as:

$$\nabla^2 g(\bar{\mathbf{r}}) = \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial x^2} + \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial y^2} + \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial z^2}$$

The scalar Laplacian can likewise be expressed in **cylindrical** and **spherical** coordinates; results given on **page 53** of your book.

The Vector Laplacian

The vector Laplacian, denoted as $\nabla^2 \mathbf{A}(\bar{r})$, both **operates** on a vector field and **results** in a vector field, and is defined as:

$$\nabla^2 \mathbf{A}(\bar{r}) \doteq \nabla\left(\nabla \cdot \mathbf{A}(\bar{r})\right) - \nabla \times \nabla \times \mathbf{A}(\bar{r})$$

Q: *Yikes! Why the heck is this* ***mess*** *referred to as the* ***Laplacian*** *?!?*

A: If we evaluate the above expression for a vector expressed in the **Cartesian** coordinate system, we find that the vector Laplacian is:

$$\nabla^2 \mathbf{A}(\bar{r}) = \nabla^2 A_x(\bar{r})\hat{a}_x + \nabla^2 A_y(\bar{r})\hat{a}_y + \nabla^2 A_z(\bar{r})\hat{a}_z$$

In other words, we evaluate the vector Laplacian by evaluating the **scalar** Laplacian of each Cartesian **scalar** component!

However, expressing the vector Laplacian in the **cylindrical** or **spherical** coordinate systems is **not** so straightforward—use instead the **definition** shown above!