The Magnetic Vector Potential

From the magnetic form of Gauss’s Law \( \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \), it is evident that the magnetic flux density \( \mathbf{B}(\mathbf{r}) \) is a solenoidal vector field.

Recall that a solenoidal field is the curl of some other vector field, e.g.:

\[
\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})
\]

**Q:** The magnetic flux density \( \mathbf{B}(\mathbf{r}) \) is the curl of what vector field??

**A:** The magnetic vector potential \( \mathbf{A}(\mathbf{r}) \)!

The curl of the magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) is equal to the magnetic flux density \( \mathbf{B}(\mathbf{r}) \):

\[
\nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r})
\]

where:
magnetic vector potential $\mathbf{A}(\mathbf{r})$ \[
\text{[Webers/meter]}
\]

Vector field $\mathbf{A}(\mathbf{r})$ is called the magnetic vector potential because of its analogous function to the electric scalar potential $V(\mathbf{r})$.

An electric field can be determined by taking the gradient of the electric potential, just as the magnetic flux density can be determined by taking the curl of the magnetic potential:

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Yikes! We have a big problem!

There are actually (infinitely) many vector fields $\mathbf{A}(\mathbf{r})$ whose curl will equal an arbitrary magnetic flux density $\mathbf{B}(\mathbf{r})$. In other words, given some vector field $\mathbf{B}(\mathbf{r})$, the solution $\mathbf{A}(\mathbf{r})$ to the differential equation $\nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r})$ is not unique!

But of course, we knew this!

To completely (i.e., uniquely) specify a vector field, we need to specify both its divergence and its curl.
Well, we know the curl of the magnetic vector potential $A(\vec{r})$ is equal to magnetic flux density $B(\vec{r})$. But, what is the divergence of $A(\vec{r})$ equal to? I.E.,:

$$\nabla \cdot A(\vec{r}) = ???$$

By answering this question, we are essentially defining $A(\vec{r})$.

Let’s define it in so that it makes our computations easier!

To accomplish this, we first start by writing Ampere’s Law in terms of magnetic vector potential:

$$\nabla \times B(\vec{r}) = \nabla \times \nabla \times A(\vec{r}) = \mu_0 J(\vec{r})$$

We recall from section 2-6 that:

$$\nabla \nabla \times A(\vec{r}) = \nabla \left( \nabla \cdot A(\vec{r}) \right) - \nabla^2 A(\vec{r})$$

Thus, we can simplify this statement if we decide that the divergence of the magnetic vector potential is equal to zero:

$$\nabla \cdot A(\vec{r}) = 0$$

We call this the gauge equation for magnetic vector potential. Note the magnetic vector potential $A(\vec{r})$ is therefore also a solenoidal vector field.
As a result of this gauge equation, we find:

\[ \nabla \times \nabla \times A(\vec{r}) = \nabla \left( \nabla \cdot A(\vec{r}) \right) - \nabla^2 A(\vec{r}) \]
\[ = -\nabla^2 A(\vec{r}) \]

And thus **Ampere’s Law** becomes:

\[ \nabla \times B(\vec{r}) = -\nabla^2 A(\vec{r}) = \mu_0 J(\vec{r}) \]

Note the Laplacian operator \( \nabla^2 \) is the **vector Laplacian**, as it operates on vector field \( A(\vec{r}) \).

**Summarizing**, we find the magnetostatic equations in terms of magnetic vector potential \( A(\vec{r}) \) are:

\[ \nabla \times A(\vec{r}) = B(\vec{r}) \]
\[ \nabla^2 A(\vec{r}) = -\mu_0 J(\vec{r}) \]
\[ \nabla \cdot A(\vec{r}) = 0 \]

Note that the magnetic form of Gauss’s equation results in the equation \( \nabla \cdot \nabla \times A(\vec{r}) = 0 \). **Why** don’t we include this equation in the above list?
Compare the magnetostatic equations using the magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) to the electrostatic equations using the electric scalar potential \( \mathbf{V}(\mathbf{r}) \):

\[
\begin{align*}
-\nabla \mathbf{V}(\mathbf{r}) &= \mathbf{E}(\mathbf{r}) \\
\nabla^2 \mathbf{V}(\mathbf{r}) &= -\frac{\rho_\nu(\mathbf{r})}{\varepsilon_0}
\end{align*}
\]

Hopefully, you see that the two potentials \( \mathbf{A}(\mathbf{r}) \) and \( \mathbf{V}(\mathbf{r}) \) are in many ways analogous.

For example, we know that we can determine a static field \( \mathbf{E}(\mathbf{r}) \) created by sources \( \rho_\nu(\mathbf{r}) \) either directly (from Coulomb’s Law), or indirectly by first finding potential \( \mathbf{V}(\mathbf{r}) \) and then taking its derivative (i.e., \( \mathbf{E}(\mathbf{r}) = -\nabla \mathbf{V}(\mathbf{r}) \)).

Likewise, the magnetostatic equations above say that we can determine a static field \( \mathbf{B}(\mathbf{r}) \) created by sources \( \mathbf{J}(\mathbf{r}) \) either directly, or indirectly by first finding potential \( \mathbf{A}(\mathbf{r}) \) and then taking its derivative (i.e., \( \nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}) \)).

\[
\begin{align*}
\rho_\nu(\mathbf{r}) &\Rightarrow \mathbf{V}(\mathbf{r}) \Rightarrow \mathbf{E}(\mathbf{r}) \\
\mathbf{J}(\mathbf{r}) &\Rightarrow \mathbf{A}(\mathbf{r}) \Rightarrow \mathbf{B}(\mathbf{r})
\end{align*}
\]