The Point Form Continuity Equation

Recall that the charge **enclosed** in a volume V can be determined from the **volume charge density**:

$$Q_{enc} = \iiint_V \rho_v(\bar{r})\, dv$$

If charge is **moving** (i.e., current flow), then charge density **can** be a function of **time** (i.e., $\rho_v(\bar{r},t)$). As a result, we write:

$$Q_{enc}(t) = \iiint_V \rho_v(\bar{r},t)\, dv$$

Inserting this into the **continuity equation**, we get:

$$\begin{aligned}\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} &= -\frac{d\,Q_{enc}(t)}{dt} \\ &= -\frac{d}{dt}\iiint_V \rho_v(\bar{r},t)\, dv\end{aligned}$$

where closed surface S **surrounds** volume V.

Now recall the **divergence theorem**! Using this theorem, know that:

$$\oint\!\!\oint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = \iiint_V \nabla \cdot \mathbf{J}(\bar{r})\, dv$$

Combining this with the continuity equation, we find:

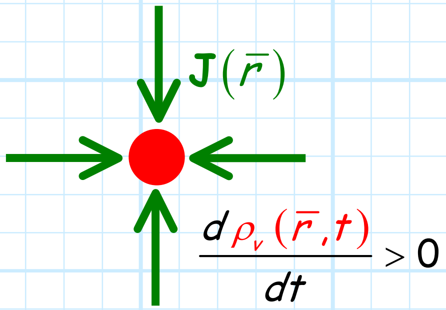
$$\iiint_V \nabla \cdot \mathbf{J}(\bar{r})\, dv = -\frac{d}{dt} \iiint_V \rho_v(\bar{r},t)\, dv$$

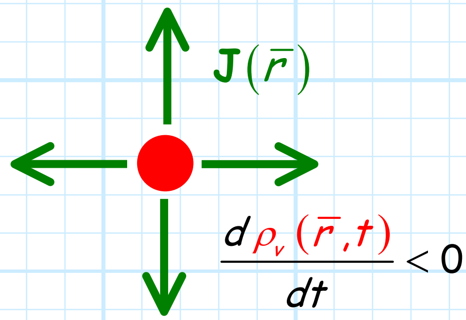
From this equation, we can conclude:

$$\nabla \cdot \mathbf{J}(\bar{r}) = -\frac{d\rho_v(\bar{r},t)}{dt}$$

This is the **point form** of the continuity equation. It says that if the **density** of charge at some point $\bar{r}$ is **increasing** with time, then **current** must be **converging** to that point.

Or, if charge density is **decreasing** with time, then current is **diverging** from point $\bar{r}$.





Current is **converging** on point, therefore charge density is **increasing**.

Current is **diverging** from point, therefore charge density is **decreasing**.

Notice that the scalar field:

$$\frac{d\rho_v(\bar{r},t)}{dt}$$

describes the **rate** at which the charge density is increasing at each and every point in the universe!

For **example**, say the divergence of $\mathbf{J}(\bar{r})$ Amps/m^2, when evaluated at some point denoted by position vector $\bar{r}_a$, is equal to 3.0:

$$-\nabla \cdot \mathbf{J}(\bar{r})\Big|_{\bar{r}=\bar{r}_a} = 3 = \frac{d\rho(\bar{r}_a,t)}{dt} \quad \frac{Amps}{m^3}$$

This means that the charge density at point $\bar{r}_a$ is increasing at a **rate** of 3 coulombs/m^3 every second!

E.G.: In **4** seconds, the charge density at $\bar{r}_a$ will **increase** by a value of **12** C/m^3.

Note the equation:

$$-\nabla \cdot \mathbf{J}(\bar{r})\Big|_{\bar{r}=\bar{r}_a} = 3 = \frac{d\,\rho(\bar{r}_a,t)}{dt}$$

is a **differential equation**. Our task is to **find** the function $\rho(\bar{r}_a,t)$, given that we know its time derivative is equal to 3.0.

The solution for this **example** can be found by **integrating** both sides of the equation (with respect to time), i.e.:

$$\rho(\bar{r}_a,t) = 3t + \rho(\bar{r}_a,t=0) \qquad \frac{C}{m^3}$$

where $\rho(\bar{r}_a,t=0)$ simply indicates the value of the charge density at point $\bar{r}_a$, at time t = 0. The value $3t$ is then the additional charge density (beyond $\rho(\bar{r}_a,t=0)$) at time t.