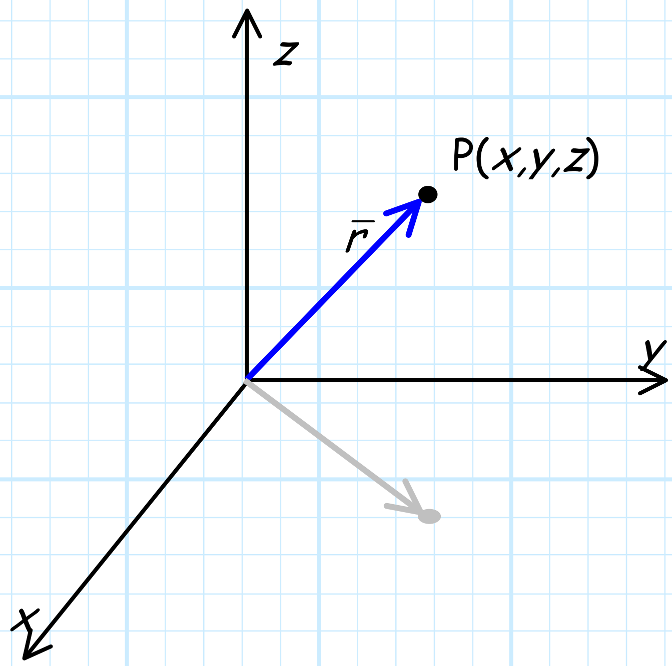
The Position Vector

Consider a point whose location in space is specified with Cartesian coordinates (e.g., P(x,y,z)). Now consider the **directed distance** (a vector quantity!) extending from the origin to this point.



This **particular** directed distance—a vector beginning at the **origin** and extending outward to a point—is a **very important** and fundamental directed distance known as the **position vector** $\bar{r}$.

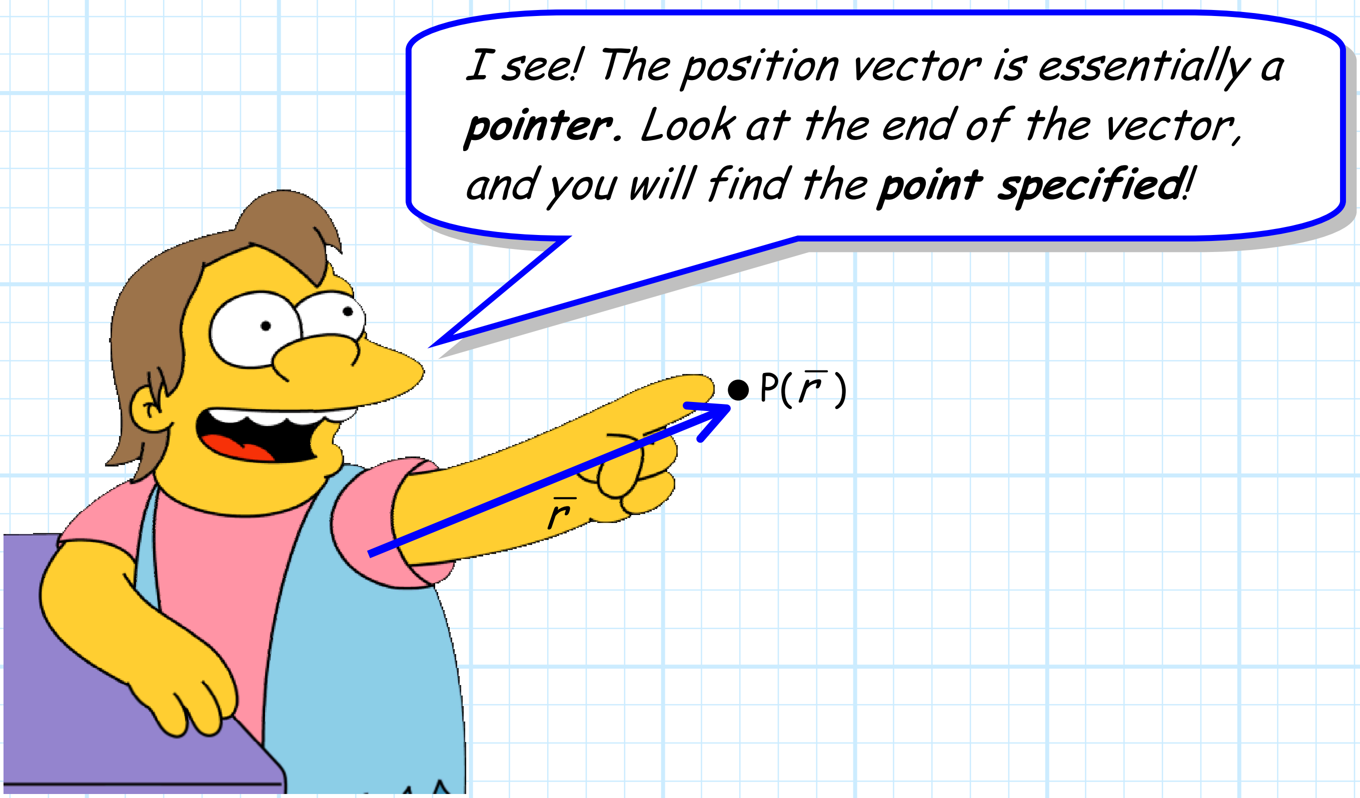
Using the **Cartesian** coordinate system, the position vector can be explicitly written as:

$$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$

* Note that given the **coordinates** of some point (e.g., x=1, y =2, z=-3), we can easily determine the corresponding **position vector** (e.g., $\bar{r} = \hat{a}_x + 2\hat{a}_y - 3\hat{a}_z$).

* Moreover, given some specific position vector (e.g., $\bar{r} = 4\hat{a}_y - 2\hat{a}_z$), we can easily determine the corresponding coordinates of that point (e.g., x=0, y=4, z=-2).

In other words, a position vector $\bar{r}$ is an alternative way to denote the location of a point in space! We can use **three coordinate values** to specify a point's location, **or** we can use a **single position vector** $\bar{r}$.



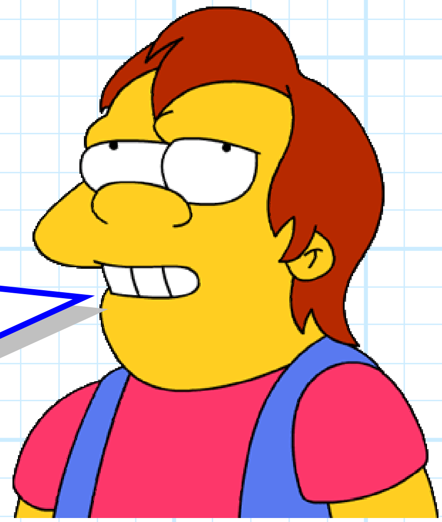
The magnitude of $\bar{r}$

Note the **magnitude** of any and all position vectors is:

$$|\bar{r}| = \sqrt{\bar{r} \cdot \bar{r}} = \sqrt{x^2 + y^2 + z^2} = r$$

The magnitude of the position vector is equal to the **coordinate value** r of the point the position vector is pointing to!

*Q: Hey, this makes **perfect sense**! Doesn't the coordinate value r have a **physical** interpretation as the **distance** between the **point** and the **origin**?*



A: That's right! The **magnitude** of a **directed distance** vector is equal to the **distance** between the two points—in this case the distance between the **specified point** and the **origin**!

Alternative forms of the position vector

Be **careful**! Although the position vector **is correctly** expressed as:

$$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$

It is **NOT CORRECT** to express the position vector as:

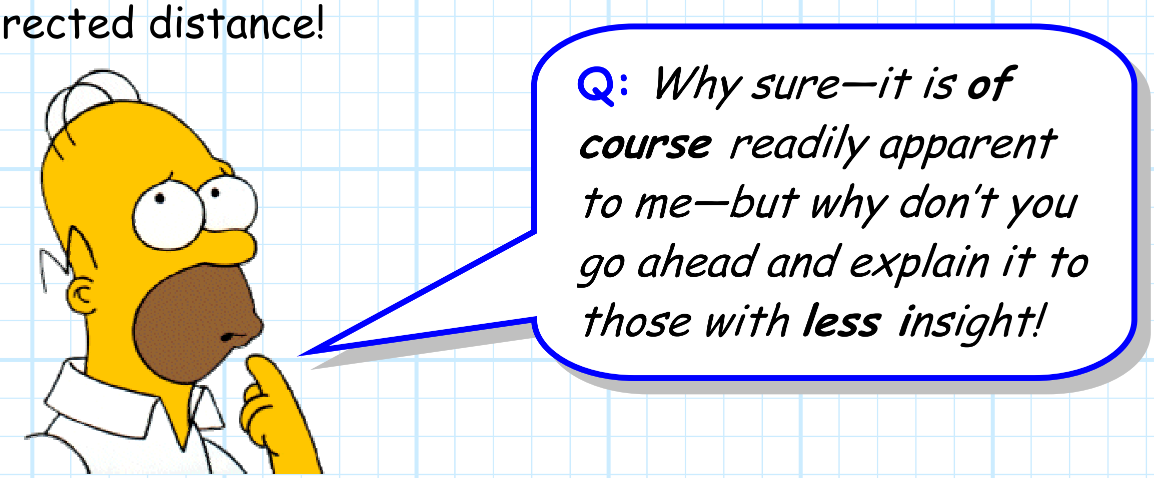
$$\bar{r} \neq \rho\, \hat{a}_p + \phi\, \hat{a}_\phi + z\, \hat{a}_z$$

nor

$$\bar{r} \neq r\, \hat{a}_r + \theta\, \hat{a}_\theta + \phi\, \hat{a}_\phi$$

NEVER, EVER express the position vector in either of these two ways!

It should be **readily apparent** that the two expression above **cannot** represent a position vector—because **neither** is even a directed distance!



A: Recall that the **magnitude** of the position vector $\bar{r}$ has units of **distance**. Thus, the **scalar components** of the position vector must **also** have units of distance (e.g., meters). The coordinates x, y, z, ρ and r **do** have units of distance, but coordinates θ and ϕ do **not**.

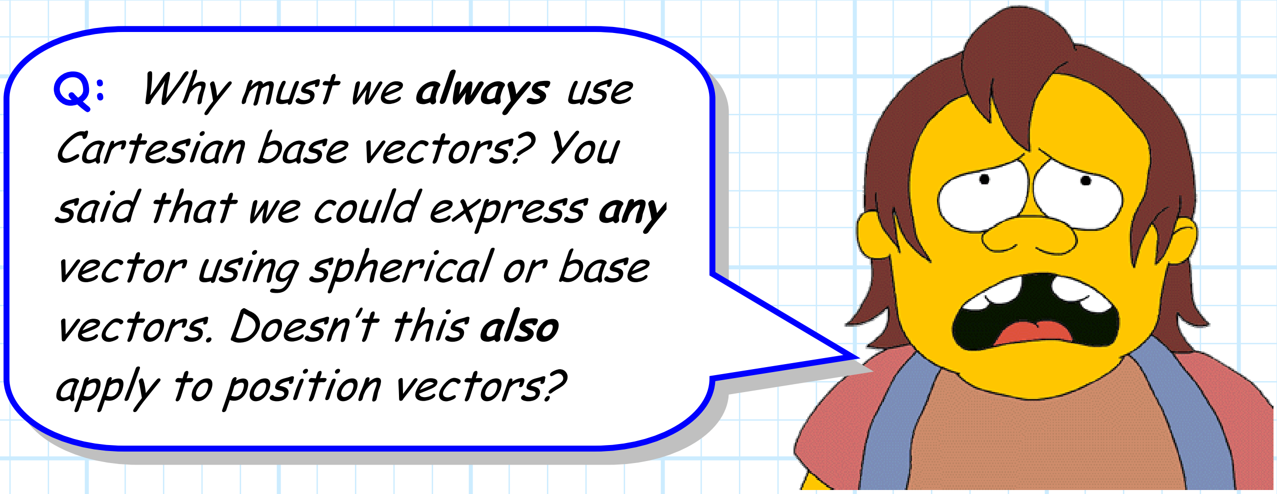
Thus, the vectors $\theta\, \hat{\mathbf{a}}_\theta$ and $\phi\, \hat{\mathbf{a}}_\phi$ **cannot** be vector components of a position vector—or for that matter, any other **directed distance**!

Instead, we can use **coordinate transforms** to show that:

$$
\begin{aligned}
\bar{r} &= x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\
&= \rho\cos\phi\,\hat{\mathbf{a}}_x + \rho\sin\phi\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\
&= r\sin\theta\cos\phi\,\hat{\mathbf{a}}_x + r\sin\theta\sin\phi\,\hat{\mathbf{a}}_y + r\cos\theta\,\hat{\mathbf{a}}_z
\end{aligned}
$$

ALWAYS use one of these three expressions of a position vector!!

Note that in **each** of the three expressions above, we use **Cartesian base vectors**. The **scalar components** can be expressed using Cartesian, cylindrical, or spherical **coordinates**, but we must always use **Cartesian base vectors**.



A: The reason we **only** use Cartesian base vectors for constructing a position vector is that Cartesian base vectors are the only base vectors whose directions are **fixed**—independent of position in space!

To see why this is important, let's go ahead and **change** the **base vectors** used to express the position vector from Cartesian to spherical or cylindrical. If we do this, we find:

$$\begin{aligned} \bar{r} &= x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z \\ &= \rho\,\hat{\mathbf{a}}_\rho + z\,\hat{\mathbf{a}}_z \\ &= r\,\hat{\mathbf{a}}_r \end{aligned}$$

Thus, the position vector expressed with the cylindrical coordinate **system** is $\bar{r} = \rho\,\hat{\mathbf{a}}_\rho + z\,\hat{\mathbf{a}}_z$, while with the spherical coordinate **system** we get $\bar{r} = r\,\hat{\mathbf{a}}_r$.

The **problem** with these two expressions is that the direction of base vectors $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_r$ are **not constant**. Instead, they themselves are vector fields—their direction is a function of position!

Thus, an expression such as $\bar{r} = 6\,\hat{\mathbf{a}}_r$ does not explicitly define a point in space, as we do not know in what **direction** base vector $\hat{\mathbf{a}}_r$ is pointing! The expression $\bar{r} = 6\,\hat{\mathbf{a}}_r$ does tell us that the coordinate r=6, but how do we determine what the values of coordinates θ or ϕ are? (*answer: we can't!*)

Compare this to the expression:

$$\bar{r} = \hat{a}_x + 2\,\hat{a}_y - 3\,\hat{a}_z$$

Here, the point described by the position vector is **clear** and unambiguous. This position vector identifies the point $P(x=1, y=2, z=-3)$.

Lesson learned: **Always** express a position vector using **Cartesian base vectors** (see box on previous page)!