The Solenoidal Vector Field

1. We of course recall that a conservative vector field $\mathbf{C}(\mathbf{r})$ can be identified from its curl, which is always equal to zero:

$$\nabla \times \mathbf{C}(\mathbf{r}) = 0$$

Similarly, there is another type of vector field $\mathbf{S}(\mathbf{r})$, called a solenoidal field, whose divergence is always equal to zero:

$$\nabla \cdot \mathbf{S}(\mathbf{r}) = 0$$

Moreover, we find that only solenoidal vector have zero divergence! Thus, zero divergence is a test for determining if a given vector field is solenoidal.

We sometimes refer to a solenoidal field as a divergenceless field.

2. Recall that another characteristic of a conservative vector field is that it can be expressed as the gradient of some scalar field (i.e., $\mathbf{C}(\mathbf{r}) = \nabla g(\mathbf{r})$).
Solenoidal vector fields have a similar characteristic! Every solenoidal vector field can be expressed as the curl of some other vector field (say $A(\vec{r})$).

$$S(\vec{r}) = \nabla \times A(\vec{r})$$

Additionally, we find that only solenoidal vector fields can be expressed as the curl of some other vector field. Note this means that:

The curl of any vector field always results in a solenoidal field!

Note if we combine these two previous equations, we get a vector identity:

$$\nabla \cdot \nabla \times A(\vec{r}) = 0$$

a result that is always true for any and every vector field $A(\vec{r})$.

Note this result is analogous to the identify derived from conservative fields:

$$\nabla \times \nabla g(\vec{r}) = 0$$

for all scalar fields $g(\vec{r})$. 
3. Now, let's recall the divergence theorem:

$$\iiint_{V} \nabla \cdot \mathbf{A}(\mathbf{r}) \, dV = \iint_{S} \mathbf{A}(\mathbf{r}) \cdot \mathbf{n} \, dS$$

If the vector field \( \mathbf{A}(\mathbf{r}) \) is solenoidal, we can write this theorem as:

$$\iiint_{V} \nabla \cdot \mathbf{S}(\mathbf{r}) \, dV = \iint_{S} \mathbf{S}(\mathbf{r}) \cdot \mathbf{n} \, dS$$

But of course, the divergence of a solenoidal field is zero \( (\nabla \cdot \mathbf{S}(\mathbf{r}) = 0) \)!

As a result, the left side of the divergence theorem is zero, and we can conclude that:

$$\iint_{S} \mathbf{S}(\mathbf{r}) \cdot \mathbf{n} \, dS = 0$$

In other words the surface integral of any and every solenoidal vector field across a closed surface is equal to zero.

Note this result is analogous to evaluating a line integral of a conservative field over a closed contour

$$\oint_{C} \mathbf{C}(\mathbf{r}) \cdot d\ell = 0$$
Let's summarize what we know about solenoidal vector fields:

1. Every solenoidal field can be expressed as the curl of some other vector field.

2. The curl of any and all vector fields always results in a solenoidal vector field.

3. The surface integral of a solenoidal field across any closed surface is equal to zero.

4. The divergence of every solenoidal vector field is equal to zero.

5. The divergence of a vector field is zero only if it is solenoidal.