The Surface Integral

An important type of vector integral that is often quite useful for solving physical problems is the **surface integral**:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

Some important things to note:

* The integrand is a **scalar** function.

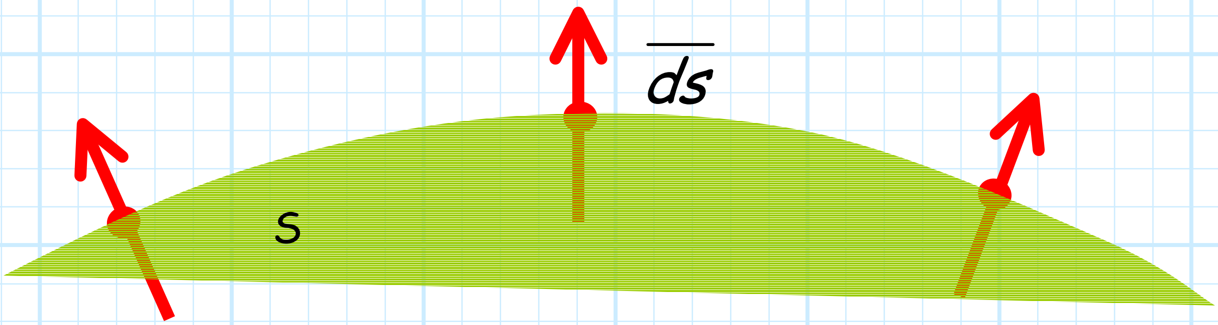
* The integration is over **two** dimensions.

* The **surface** S is an arbitrary two-dimensional surface in a three-dimensional space.

* The position vector $\bar{r}_s$ denotes only those points that lie on surface S. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points on this surface.

Q: *How are differential surface vector $\overline{ds}$ and surface S related?*

A: The differential vector $\overline{ds}$ describes a differential surface area at every point on S.



As a result, the differential surface vector $\overline{ds}$ is **normal** (i.e., orthogonal) to surface S at every point on S.

Q: *So what does the scalar integrand $\mathbf{A}(\bar{r}_s)\cdot\overline{ds}$ mean? What is it that we are actually integrating?*

A: Essentially, the surface integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\bar{r})$ at **each and every point** on surface S. This scalar component of vector field $\mathbf{A}(\bar{r})$ is the projection of $\mathbf{A}(\bar{r}_s)$ onto a direction perpendicular (i.e., normal) to the surface S.

First, I must point out that the notation $\mathbf{A}(\bar{r}_s)$ is **non-standard**. Typically, the vector field in the surface integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_s)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at points that lie on surface S, and the points that lie on surface S are denoted by position vector $\bar{r}_s$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do **not** lie on the surface (which is just about all of them!) have **no effect** on the integration. The integral **only** depends on the value of the vector field as we move over surface S—we denote these values as $\mathbf{A}(\bar{r}_s)$.

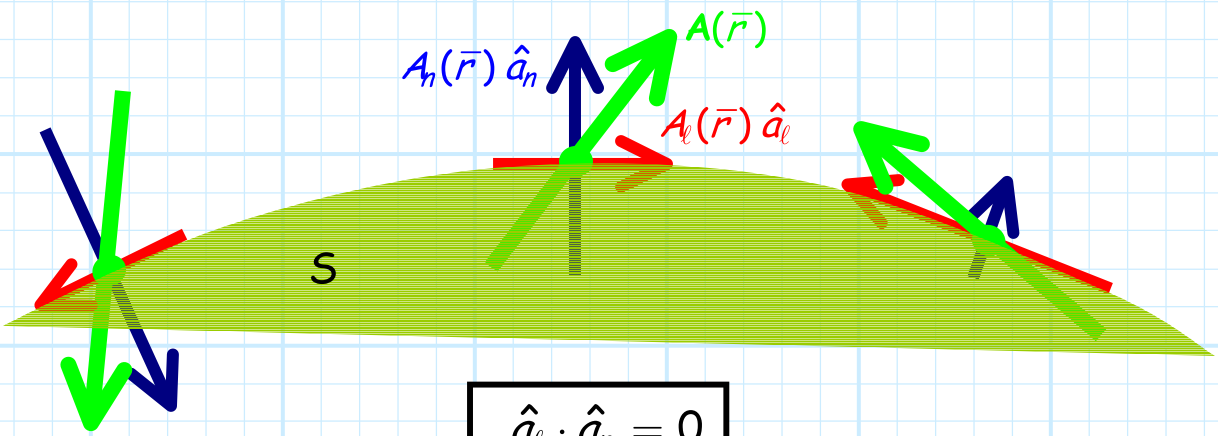
Moreover, the surface integral depends on **only one component** of $\mathbf{A}(\bar{r}_s)$!

Q: *On just what component of* $\mathbf{A}(\bar{r}_s)$ ***does*** *the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$ --we see it involves the **dot** product! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_s)$ onto the differential vector $\overline{ds}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_s)$ that lies in the direction of $\overline{ds}$ --and $\overline{ds}$ **always** points in the direction orthogonal to surface S!

To help see this, first note that every vector $\mathbf{A}(\bar{r}_s)$ can be written in terms of a component tangential to the surface (i.e, $A_\ell(\bar{r}_s)\,\hat{a}_\ell$), and a component that is **normal** (i.e., orthogonal) to the surface (i.e, $A_n(\bar{r}_s)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_s) = A_\ell(\bar{r}_s)\,\hat{a}_\ell + A_n(\bar{r}_s)\,\hat{a}_n$$



$$\hat{a}_\ell \cdot \hat{a}_n = 0$$

We note that the differential surface vector $\overline{ds}$ can be written in terms of its magnitude ($\left|\overline{ds}\right|$) and direction ($\hat{a}_n$) as:

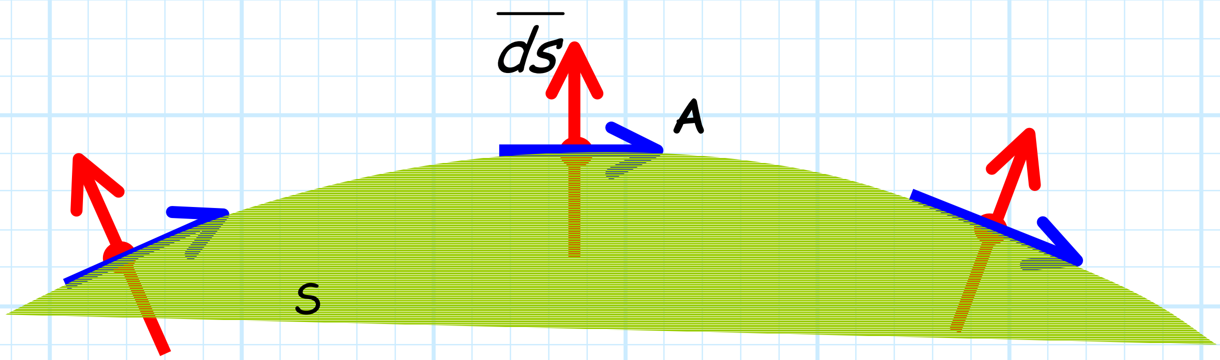
$$\overline{ds} = \hat{a}_n \left|\overline{ds}\right|$$

For example, for $\overline{ds_r} = \hat{a}_r\, r^2 \sin\theta\, d\theta d\phi$, we can say $\left|\overline{ds_r}\right| = r^2 \sin\theta\, d\theta\, d\phi$ and $\hat{a}_n = \hat{a}_r$.

As a result we can write:

$$\begin{aligned}
\iint_S \mathbf{A}(\bar{r}) \cdot \overline{ds} &= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell + A_n(\bar{r})\, \hat{a}_n \right] \cdot \overline{ds} \\
&= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell + A_n(\bar{r})\, \hat{a}_n \right] \cdot\, \hat{a}_n \left| \overline{ds} \right| \\
&= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell \cdot \hat{a}_n + A_n(\bar{r})\, \hat{a}_n \cdot \hat{a}_n \right] \left| \overline{ds} \right| \\
&= \iint_S A_n(\bar{r}) \left| \overline{ds} \right|
\end{aligned}$$

Note if vector field $\mathbf{A}(\bar{r})$ is **tangential** to the surface at every point, then the resulting surface integral will be **zero**.



Although S represents **any** surface, no matter how **complex** or **convoluted**, we will study only **basic** surfaces. In other words, $\overline{ds}$ will correspond to one of the differential surface vectors from Cartesian, cylindrical, or spherical coordinate systems.