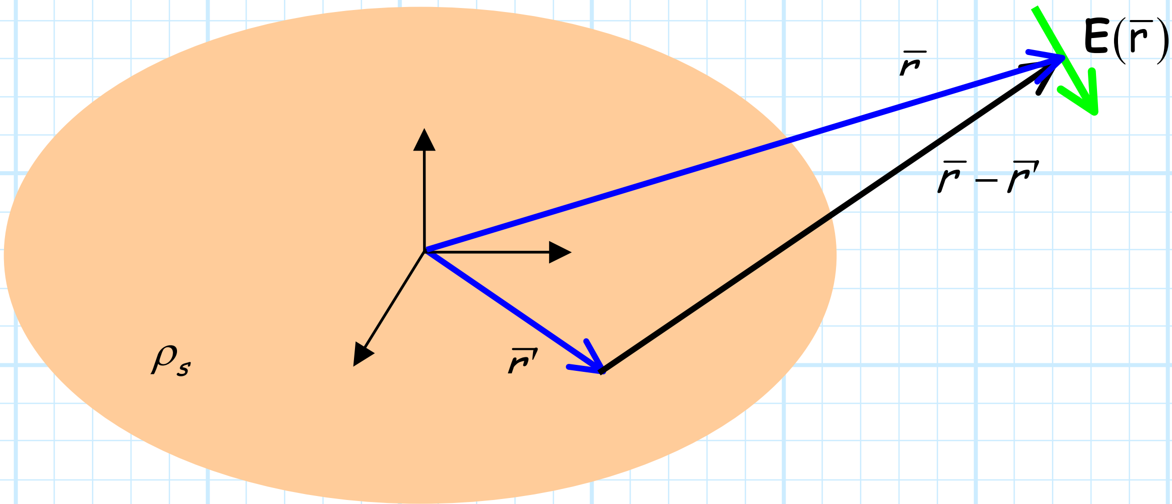
The Uniform Disk of Charge

Consider a **disk** radius *a*, centered at the origin, and lying entirely on the $z=0$ plane.



This disk contains **surface charge**, with density of ρ_s C/m^2. This density is **uniform** across the disk.

Let's find the **electric field** generated by this charge disk!

From **Coulomb's Law**, we know:

$$\mathbf{E}(\bar{r}) = \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}'}{\left|\bar{r} - \bar{r}'\right|^3} \, ds'$$

Step 1: Determine ds'

This disk can be described by the equation $z' = 0$. That is, every point on the disk has a cordinate value z' that is equal to zero.

This is **one** of the surfaces we examined in chapter 2. The **differential surface element** for that surface, you recall, is:

$$ds' = ds_z = \rho' \, d\rho' \, d\phi'$$

Step 2: Determine the **limits of integration** .

Note over the surface of the disk, ρ' changes from 0 to radius a, and ϕ' changes from 0 to 2π. Therefore:

$$0 < \rho' < a \qquad\qquad 0 < \phi' < 2\pi$$

Step 3: Determine **vector** $\bar{r}-\bar{r}'$.

We know that $z' = 0$ for all charge, therefore we can write:

$$\begin{aligned}\bar{r} - \bar{r}' &= \left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) - \left(x'\,\hat{a}_x + y'\,\hat{a}_y + z'\,\hat{a}_z\right)\\ &= \left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) - \left(x'\,\hat{a}_x + y'\,\hat{a}_y\right)\\ &= \left(x - x'\right)\hat{a}_x + \left(y - y'\right)\hat{a}_y + z\,\hat{a}_z\end{aligned}$$

Since the primed coordinates in ds' are expressed in **cylindrical** coordinates, we convert the coordinates to get:

$$\begin{aligned}\bar{r}-\bar{r}' &= \left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) - \left(x'\,\hat{a}_x + y'\,\hat{a}_y\right) \\ &= \left(x - x'\right)\,\hat{a}_x + \left(y - y'\right)\hat{a}_y + z\,\hat{a}_z \\ &= \left(x - \rho'\cos\phi'\right)\hat{a}_x + \left(y - \rho'\sin\phi'\right)\hat{a}_y + z\,\hat{a}_z\end{aligned}$$

Step 4: Determine $\left|\bar{r} - \bar{r}'\right|^3$

We find that:

$$\left|\bar{r} - \bar{r}'\right|^3 = \left[\left(x - \rho'\cos\phi'\right)^2 + \left(y - \rho'\sin\phi'\right)^2 + z^2\right]^{3/2}$$

Step 5: Time to integrate !

$$\begin{aligned}\mathbf{E}(\bar{r}) &= \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0}\frac{\bar{r} - \bar{r}'}{\left|\bar{r} - \bar{r}'\right|^3}\,ds' \\ &= \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{a}\frac{\left(x - \rho'\cos\phi'\right)\hat{a}_x + \left(y - \rho'\sin\phi'\right)\hat{a}_y + z\,\hat{a}_z}{\left[\left(x - \rho'\cos\phi'\right)^2 + \left(y - \rho'\sin\phi'\right)^2 + z^2\right]^{3/2}}\,\rho'd\rho'd\phi'\end{aligned}$$

Yikes! What a **mess**! To **simplify** our integration let's determine the electric field $\mathbf{E}(\bar{r})$ along the ***z*-axis** only. In other words, set $x = 0$ and $y = 0$.

$$\mathbf{E}(x=0,y=0,z) = \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3}\, ds'$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^a \frac{(0-\rho'\cos\phi')\hat{a}_x + (0-\rho'\sin\phi')\hat{a}_y - z\,\hat{a}_z}{\left[(0-\rho'\cos\phi')^2 + (0-\rho'\sin\phi')^2 + z^2\right]^{3/2}}\, \rho' d\rho' d\phi'$$

$$= \frac{-\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^a \frac{(\rho'\cos\phi')\hat{a}_x + (\rho'\sin\phi')\hat{a}_y - z\,\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}}\, \rho' d\rho' d\phi'$$

$$= \frac{\rho_s}{4\pi\varepsilon_0}\hat{a}_x \int_0^{2\pi}\int_0^a \frac{(\rho'\cos\phi')\,\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$+ \frac{-\rho_s}{4\pi\varepsilon_0}\hat{a}_y \int_0^{2\pi}\int_0^a \frac{(\rho'\sin\phi')\,\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$+ \frac{-\rho_s}{4\pi\varepsilon_0}\hat{a}_z \int_0^{2\pi}\int_0^a \frac{z\ \rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

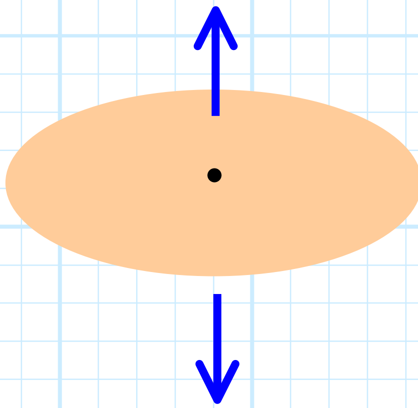
Note that since:

$$\int_0^{2\pi} \sin\phi\, d\phi = 0 = \int_0^{2\pi} \cos\phi\, d\phi$$

The first two terms (E_x and E_y) are equal to zero. Integrating the last term, we get:

$$\mathbf{E}(x=0,y=0,z) = \begin{cases} \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z \left[1 - \dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z > 0 \\[2ex] \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z \left[-1 - \dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z < 0 \end{cases}$$

From this expression, we can conclude **two** things. The first is that **above** the disk (z > 0), the electric field points in the direction $\hat{a}_z$, and below the disk (z < 0), it points in the direction $-\hat{a}_z$.



What a surprise (not)! The electric field **points away** from the charge. It appears to be **diverging** from the charged disk (as predicted by Gauss's Law).

Likewise, it is evident that as we move further and **further from** the disk, the electric field will **diminish**. In fact, as distance z goes to **infinity**, the magnitude of the electric field approaches **zero**. This of course is similar to the **point** or **line** charge; as we move an infinite distance away, the electric field diminishes to **nothing**.