The Uniform, Infinite Line Charge

Consider an infinite line of charge lying along the z-axis. The charge density along this line is a constant value of \( \rho \) C/m.

**Q:** What electric field \( \mathbf{E}(\mathbf{r}) \) is produced by this charge distribution?

**A:** Apply Coulomb's Law!

We know that for a line charge distribution that:

\[
\mathbf{E}(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{c} \frac{\mathbf{r} - \mathbf{r}'}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3} d' \]

\[ d' \]

\[
\rho(\mathbf{r}')
\]

\[
\mathbf{r} - \mathbf{r}'
\]

\[
4\pi\varepsilon_0
\]

\[
|\mathbf{r} - \mathbf{r}'|^3
\]

\[
d'
\]
**Q:** Yikes! How do we evaluate this integral?

**A:** Don't panic! You know how to evaluate this integral. Let's break up the process into smaller steps.

**Step 1:** Determine $d\ell'$

The differential element $d\ell'$ is just the magnitude of the differential line element we studied in chapter 2 (i.e., $d\ell' = |d\ell'|$). As a result, we can easily integrate over any of the seven contours we discussed in chapter 2.

The contour in this problem is one of those! It is a line parallel to the z-axis, defined as $x' = 0$ and $y' = 0$. As a result, we use for $d\ell'$:

$$d\ell' = |\hat{z} dz'| = dz'$$

**Step 2:** Determine the limits of integration

This is easy! The line charge is infinite. Therefore, we integrate from $z' = -\infty$ to $z' = \infty$.

**Step 3:** Determine the vector $\vec{r} - \vec{r}'$.

Since for all charge $x' = 0$ and $y' = 0$, we find:

$$\vec{r} - \vec{r}' = (x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) - (x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z)
= (x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) - z'\hat{a}_z
= x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z$$
**Step 4:** Determine the scalar $|\vec{r} - \vec{r}'|^3$

Since $|\vec{r} - \vec{r}'| = \sqrt{x^2 + y^2 + (z - z')^2}$, we find:

$$|\vec{r} - \vec{r}'|^3 = \left[ x^2 + y^2 + (z - z')^2 \right]^{3/2}$$

**Step 5:** Time to integrate!

$$E(\vec{r}) = \int \frac{\rho_\ell(\vec{r}')}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, d\ell'$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \rho_\ell \frac{x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z}{\left[ x^2 + y^2 + (z - z')^2 \right]^{3/2}} \, dz'$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z}{\left[ x^2 + y^2 + (z - z')^2 \right]^{3/2}} \, dz'$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{2}{x^2 + y^2} \, dz'$$

$$= \frac{\rho_\ell}{2\pi\varepsilon_0} \frac{x\hat{a}_x + y\hat{a}_y}{x^2 + y^2}$$
This result, however, is best expressed in **cylindrical coordinates**:

\[
\frac{x \hat{a}_x + y \hat{a}_y}{x^2 + y^2} = \frac{\rho \cos \phi \hat{a}_x + \rho \sin \phi \hat{a}_y}{\rho^2}
\]

\[
= \frac{\cos \phi \hat{a}_x + \sin \phi \hat{a}_y}{\rho}
\]

And with cylindrical **base vectors**:

\[
\frac{\cos \phi \hat{a}_x + \sin \phi \hat{a}_y}{\rho} = \frac{1}{\rho} \left( \cos \phi \hat{a}_x \cdot \hat{a}_\rho + \sin \phi \hat{a}_y \cdot \hat{a}_\rho \right) \hat{a}_\rho
\]

\[
+ \frac{1}{\rho} \left( \cos \phi \hat{a}_x \cdot \hat{a}_\phi + \sin \phi \hat{a}_y \cdot \hat{a}_\phi \right) \hat{a}_\phi
\]

\[
+ \frac{1}{\rho} \left( \cos \phi \hat{a}_x \cdot \hat{a}_z + \sin \phi \hat{a}_y \cdot \hat{a}_z \right) \hat{a}_z
\]

\[
= \frac{1}{\rho} \left( \cos^2 \phi + \sin^2 \phi \right) \hat{a}_\rho
\]

\[
+ \frac{1}{\rho} \left( -\cos \phi \sin \phi + \sin \phi \cos \phi \right) \hat{a}_\phi
\]

\[
+ \frac{1}{\rho} \left( \cos \phi (0) + \sin \phi (0) \right) \hat{a}_z
\]

\[
= \hat{a}_\rho
\]
As a result, we can write the electric field produced by an infinite line charge with constant density $\rho$, as:

$$E(\vec{r}) = \frac{\rho}{2\pi\varepsilon_0} \frac{\hat{a}_\rho}{\rho}$$

Note what this means. Recall unit vector $\hat{a}_\rho$ is the direction that points away from the z-axis. In other words, the electric field produced by the uniform line charge points away from the line charge, just like the electric field produced by a point charge likewise points away from the charge.

It is apparent that the electric field in the static case appears to diverge from the location of the charge. And, this is exactly what Maxwell’s equations (Gauss’s Law) says will happen! i.e.,:

$$\nabla \cdot E(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

Note the magnitude of the electric field is proportional to $1/\rho$, therefore the electric field diminishes as we get further from the line charge. Note however, the electric field does not diminish as quickly as that generated by a point charge. Recall in that case, the magnitude of the electric field diminishes as $1/r^2$. 