The Uniform, Infinite Line Charge

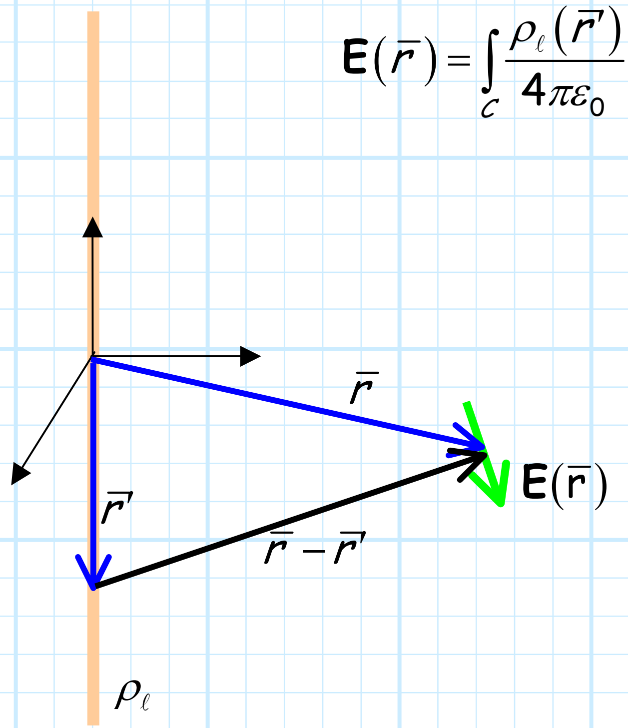
Consider an **infinite** line of charge lying along the *z*-axis. The charge density along this line is a **constant** value of ρ_ℓ C/m.

Q: *What electric field* $\mathbf{E}(\bar{\mathbf{r}})$ *is produced by* ***this*** *charge distribution?*

A: Apply **Coulomb's Law**!

We know that for a **line charge** distribution that:

$$\mathbf{E}(\bar{r}) = \int_C \frac{\rho_\ell(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}'}{\left|\bar{r} - \bar{r}'\right|^3} d\ell'$$



Q: *Yikes! How do we evaluate **this** integral?*

A: Don't panic! **You** know how to evaluate this integral. Let's break up the process into **smaller steps**.

Step 1: Determine $d\ell'$

The differential element $d\ell'$ is just the **magnitude** of the differential line element we studied in chapter 2 (i.e., $d\ell' = \left|\overline{d\ell'}\right|$). As a result, we can easily integrate over **any** of the seven contours we discussed in chapter 2.

The contour in this problem **is** one of those! It is a line parallel to the *z*-axis, defined as x'=0 and y'=0. As a result, we use for $d\ell'$:

$$d\ell' = \left|\hat{a}_z\, dz'\right| = dz'$$

Step 2: Determine the **limits of integration**

This is easy! The line charge is **infinite**. Therefore, we integrate from $z' = -\infty$ to $z' = \infty$.

Step 3: Determine the **vector** $\bar{r} - \bar{r}'$.

Since for all charge x'= 0 and y'=0, we find:

$$\begin{aligned}\bar{r} - \bar{r}' &= \left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) - \left(x'\,\hat{a}_x + y'\,\hat{a}_y + z'\,\hat{a}_z\right) \\ &= \left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) - z'\,\hat{a}_z \\ &= x\,\hat{a}_x + y\,\hat{a}_y + \left(z - z'\right)\hat{a}_z\end{aligned}$$

Step 4: Determine the **scalar** $\left|\bar{r} - \bar{r}'\right|^3$

Since $\left|\bar{r} - \bar{r}'\right| = \sqrt{x^2 + y^2 + (z - z')^2}$, we find:

$$\left|\bar{r} - \bar{r}'\right|^3 = \left[x^2 + y^2 + (z - z')^2\right]^{3/2}$$

Step 5: Time to integrate !

$$\begin{aligned}
\mathbf{E}(\bar{r}) &= \int_C \frac{\rho_\ell(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r} - \bar{r}'}{\left|\bar{r} - \bar{r}'\right|^3} \, d\ell' \\
&= \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \rho_\ell \frac{x\,\hat{a}_x + y\,\hat{a}_y + (z - z')\,\hat{a}_z}{\left[x^2 + y^2 + (z - z')^2\right]^{3/2}} \, dz' \\
&= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{x\,\hat{a}_x + y\,\hat{a}_y + (z - z')\,\hat{a}_z}{\left[x^2 + y^2 + (z - z')^2\right]^{3/2}} \, dz' \\
&= \frac{\rho_\ell\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{\left[x^2 + y^2 + (z - z')^2\right]^{3/2}} \\
&\quad + \frac{\rho_\ell\,\hat{a}_z}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{(z - z')\,dz'}{\left[x^2 + y^2 + (z - z')^2\right]^{3/2}} \\
&= \frac{\rho_\ell\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{4\pi\varepsilon_0} \frac{2}{x^2 + y^2} + 0 \\
&= \frac{\rho_\ell}{2\pi\varepsilon_0} \frac{\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{x^2 + y^2}
\end{aligned}$$

This result, however, is best expressed in **cylindrical coordinates**:

$$\frac{x\,\hat{a}_x + y\,\hat{a}_y}{x^2+y^2} = \frac{\rho\cos\phi\,\hat{a}_x + \rho\sin\phi\,\hat{a}_y}{\rho^2}$$

$$= \frac{\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y}{\rho}$$

And with cylindrical **base vectors**:

$$\begin{aligned}\frac{\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y}{\rho} &= \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_\rho + \sin\phi\,\hat{a}_y\cdot\hat{a}_\rho\right)\hat{a}_\rho \\ &\quad + \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_\phi + \sin\phi\,\hat{a}_y\cdot\hat{a}_\phi\right)\hat{a}_\phi \\ &\quad + \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_z + \sin\phi\,\hat{a}_y\cdot\hat{a}_z\right)\hat{a}_z \\ &= \frac{1}{\rho}\left(\cos^2\phi + \sin^2\phi\right)\hat{a}_\rho \\ &\quad + \frac{1}{\rho}\left(-\cos\phi\sin\phi + \sin\phi\cos\phi\right)\hat{a}_\phi \\ &\quad + \frac{1}{\rho}\left(\cos\phi(0) + \sin\phi(0)\right)\hat{a}_z \\ &= \frac{\hat{a}_\rho}{\rho}\end{aligned}$$

As a result, we can write the **electric field** produced by an **infinite line charge** with constant density ρ_ℓ as:

$$\mathbf{E}(\bar{r}) = \frac{\rho_\ell}{2\pi\varepsilon_0}\frac{\hat{a}_\rho}{\rho}$$

Note what this means. Recall unit vector $\hat{a}_\rho$ is the direction that **points away from** the z-axis. In other words, the electric field produced by the uniform line charge points away from the line charge, just like the electric field produced by a point charge likewise points away from the charge.

It is apparent that the electric field in the static case appears to **diverge** from the location of the charge. And, this is exactly what Maxwell's equations (**Gauss's Law**) says will happen ! i.e.,:

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

Note the **magnitude** of the electric field is proportional to $1/\rho$, therefore the electric field **diminishes** as we get further from the line charge. Note however, the electric field does not diminish as **quickly** as that generated by a point charge. Recall in that case, the magnitude of the electric field diminishes as $1/r^2$.