Vector Algebra using Orthonormal Base Vectors

A: Actually, it makes things much simpler. The evaluation of vector operations such as addition, subtraction, multiplication, dot product, and cross product all become straightforward if all vectors are expressed using the same set of base vectors.

Consider two vectors \( \mathbf{A} \) and \( \mathbf{B} \), each expressed using the same set of base vectors \( \hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z \):

\[
\mathbf{A} = A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z
\]

\[
\mathbf{B} = B_x \hat{\mathbf{a}}_x + B_y \hat{\mathbf{a}}_y + B_z \hat{\mathbf{a}}_z
\]

Q: Just why do we express a vector in terms of 3 orthonormal base vectors? Doesn’t this just make things even more complicated??
1. **Addition and Subtraction**

If we add these two vectors together, we find:

\[ \mathbf{A} + \mathbf{B} = (A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z) + (B_x \hat{\mathbf{a}}_x + B_y \hat{\mathbf{a}}_y + B_z \hat{\mathbf{a}}_z) = \]

\[ = A_x \hat{\mathbf{a}}_x + B_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + B_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z + B_z \hat{\mathbf{a}}_z = \]

\[ = (A_x + B_x) \hat{\mathbf{a}}_x + (A_y + B_y) \hat{\mathbf{a}}_y + (A_z + B_z) \hat{\mathbf{a}}_z \]

In other words, each component of the sum of two vectors is equal to the sum of each component.

Similarly, we find for subtraction:

\[ \mathbf{A} - \mathbf{B} = (A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z) - (B_x \hat{\mathbf{a}}_x + B_y \hat{\mathbf{a}}_y + B_z \hat{\mathbf{a}}_z) = \]

\[ = A_x \hat{\mathbf{a}}_x - B_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y - B_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z - B_z \hat{\mathbf{a}}_z = \]

\[ = (A_x - B_x) \hat{\mathbf{a}}_x + (A_y - B_y) \hat{\mathbf{a}}_y + (A_z - B_z) \hat{\mathbf{a}}_z \]

2. **Vector/Scalar Multiplication**

Say we multiply a scalar \( a \) and a vector \( \mathbf{B} \), i.e., \( a \mathbf{B} \):
\[ a \mathbf{B} = a \left( B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}} \right) \\
= aB_x \hat{\mathbf{x}} + aB_y \hat{\mathbf{y}} + aB_z \hat{\mathbf{z}} \\
= (aB_x) \hat{\mathbf{x}} + (aB_y) \hat{\mathbf{y}} + (aB_z) \hat{\mathbf{z}} \]

In other words, each component of the product of a scalar and a vector are equal to the product of the scalar and each component.

3. Dot Product

Say we take the dot product of \( \mathbf{A} \) and \( \mathbf{B} \):

\[ \mathbf{A} \cdot \mathbf{B} = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\
= A_x \hat{\mathbf{x}} \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\
+ A_y \hat{\mathbf{y}} \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\
+ A_z \hat{\mathbf{z}} \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\
= A_x B_x (\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) + A_x B_y (\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) + A_x B_z (\hat{\mathbf{x}} \cdot \hat{\mathbf{z}}) \\
+ A_y B_x (\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}) + A_y B_y (\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}) + A_y B_z (\hat{\mathbf{y}} \cdot \hat{\mathbf{z}}) \\
+ A_z B_x (\hat{\mathbf{z}} \cdot \hat{\mathbf{x}}) + A_z B_y (\hat{\mathbf{z}} \cdot \hat{\mathbf{y}}) + A_z B_z (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \]

**Q:** I thought this was supposed to make things easier!?!?

**A:** Be patient! Recall that these are orthonormal base vectors, therefore:

\[ \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1 \quad \text{and} \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0 \]
As a result, our **dot product** expression reduces to this simple expression:

\[
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z
\]

We can apply this to the expression for determining the **magnitude** of a vector:

\[
|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2
\]

Therefore:

\[
|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}
\]

For example, consider a previous handout, where we expressed a vector using two different sets of basis vectors:

\[
\mathbf{A} = 2.0 \hat{a}_x + 1.5 \hat{a}_y
\]

or,

\[
\mathbf{A} = 2.5 \hat{b}_y
\]

Therefore, the magnitude of \( \mathbf{A} \) is determined to be:
\[ |A| = \sqrt{1.5^2 + 2.0^2} = \sqrt{6.25} = 2.5 \]

or,

\[ |A| = \sqrt{2.5^2} = \sqrt{6.25} = 2.5 \]

**Q:** Hey! We get the *same* answer from both expressions; is this a *coincidence*?

**A:** No! Remember, both expressions represent the *same* vector, only using different sets of base vectors. The magnitude of vector \( A \) is 2.5, regardless of how we choose to express \( A \).

4. **Cross Product**

Now let's take the cross product \( A \times B \):

\[
A \times B = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \times (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z)
\]

\[
= A_x \hat{a}_x \times (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z)
\]

\[
+ A_y \hat{a}_y \times (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z)
\]

\[
+ A_z \hat{a}_z \times (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z)
\]

\[
= A_x B_x (\hat{a}_x \times \hat{a}_x) + A_x B_y (\hat{a}_x \times \hat{a}_y) + A_x B_z (\hat{a}_x \times \hat{a}_z)
\]

\[
+ A_y B_x (\hat{a}_y \times \hat{a}_x) + A_y B_y (\hat{a}_y \times \hat{a}_y) + A_y B_z (\hat{a}_y \times \hat{a}_z)
\]

\[
+ A_z B_x (\hat{a}_z \times \hat{a}_x) + A_z B_y (\hat{a}_z \times \hat{a}_y) + A_z B_z (\hat{a}_z \times \hat{a}_z)
\]
Remember, we know that:

\[ \hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0 \]

also, since base vectors form a right-handed system:

\[ \hat{a}_x \times \hat{a}_y = \hat{a}_z \quad \hat{a}_y \times \hat{a}_z = \hat{a}_x \quad \hat{a}_z \times \hat{a}_x = \hat{a}_y \]

Remember also that \( A \times B = -(B \times A) \), therefore:

\[ \hat{a}_y \times \hat{a}_x = -\hat{a}_z \quad \hat{a}_z \times \hat{a}_y = -\hat{a}_x \quad \hat{a}_x \times \hat{a}_z = -\hat{a}_y \]

Combining all the equations above, we get:

\[ A \times B = (A_x B_z - A_z B_y) \hat{a}_x + (A_z B_x - A_x B_z) \hat{a}_y + (A_x B_y - A_y B_x) \hat{a}_z \]

5. **Triple Product**

Combining the results of the dot product and the cross product, we find that the **triple product** can be expressed as:

\[ A \cdot (B \times C) = (A_x B_y C_z + A_y B_z C_x + A_z B_x C_y) - (A_x B_z C_y + A_y B_x C_z + A_z B_y C_x) \]
IMPORTANT NOTES:

In addition to all that we have discussed here, it is critical that you understand the following points about vector algebra using orthonormal base vectors!

* The results provided in this handout were given for Cartesian base vectors (\( \hat{a}_x, \hat{a}_y, \hat{a}_z \)). However, they are equally valid for any right-handed set of base vectors \( \hat{a}_1, \hat{a}_2, \hat{a}_3 \) (e.g., \( \hat{a}_r, \hat{a}_\theta, \hat{a}_\phi \) or \( \hat{a}_r, \hat{a}_\theta, \hat{a}_\phi \)).

* These results are algorithms for evaluating various vector algebraic operations. They are not definitions of the operations. The definitions of these operations were covered in Section 2-3.

* The scalar components \( A_x, A_y, \) and \( A_z \) represent either discrete scalar (e.g., \( A_x = 4.2 \)) or scalar field quantities (e.g., \( A_\theta = r^2 \sin \theta \cos \phi \)).