Vector Field Notation

A vector field describes a vector value at every location in space. Therefore, we can denote a vector field as $A(x,y,z)$, or $A(\rho,\phi,z)$, or $A(r,\theta,\phi)$, explicitly showing that vector quantity $A$ is a function of position, as denoted by some set of coordinates.

However, as we have emphasized before, the physical reality that vector field $A$ expresses is independent of the coordinates we use to express it. In other words, although the math may look very different, we find that:

$$A(x,y,z) = A(\rho,\phi,z) = A(r,\theta,\phi).$$

Alternatively then, we typically express a vector field as simply:

$$A(\vec{r})$$

This symbolically says everything that we need to convey; vector $A$ is a function of position—it is a vector field!

Note that the vector field notation $A(\vec{r})$ does not explicitly specify a coordinate system for expressing $A$. That’s up to you to decide!
Now, in the vector field expression $\mathbf{A}(\mathbf{r})$ we note that there are two vectors: $\mathbf{A}$ and $\mathbf{r}$. It is **ridiculously important** that you understand what each of these two vectors represents!

**Position vector $\mathbf{r}$** denotes the location in space where vector $\mathbf{A}$ is defined.

For example, consider the vector field $\mathbf{V}(\mathbf{r})$, which describes the **wind velocity** across the state of Kansas.

In this map, the **origin** has been placed at Lawrence. The **locations** of Kansas towns can thus be identified using **position vectors** (units in miles):
\[ \vec{r}_1 = -400 \hat{a}_x + 20 \hat{a}_y \quad \rightarrow \quad \text{the location of Goodland, KS} \]

\[ \vec{r}_2 = -90 \hat{a}_x + 70 \hat{a}_y \quad \rightarrow \quad \text{the location of Marysville, KS} \]

\[ \vec{r}_3 = 30 \hat{a}_x - 5 \hat{a}_y \quad \rightarrow \quad \text{the location of Fort Scott, KS} \]

\[ \vec{r}_4 = 40 \hat{a}_x - 90 \hat{a}_y \quad \rightarrow \quad \text{the location of Fort Scott, KS} \]

\[ \vec{r}_5 = -130 \hat{a}_x - 70 \hat{a}_y \quad \rightarrow \quad \text{the location of Newton, KS} \]

Evaluating the vector field \( \mathbf{V}(\vec{r}) \) at these locations provides the wind velocity at each Kansas town (units of mph).

\[ \mathbf{V}(\vec{r}_1) = 15 \hat{a}_x - 17 \hat{a}_y \quad \rightarrow \quad \text{the wind velocity in Goodland, KS} \]

\[ \mathbf{V}(\vec{r}_2) = 15 \hat{a}_x - 9 \hat{a}_y \quad \rightarrow \quad \text{the wind velocity in Marysville, KS} \]

\[ \mathbf{V}(\vec{r}_3) = 11 \hat{a}_x \quad \rightarrow \quad \text{the wind velocity in Olathe, KS} \]

\[ \mathbf{V}(\vec{r}_4) = 7 \hat{a}_x \quad \rightarrow \quad \text{the wind velocity in Fort Scott, KS} \]

\[ \mathbf{V}(\vec{r}_5) = 9 \hat{a}_x - 4 \hat{a}_y \quad \rightarrow \quad \text{the wind velocity in Newton, KS} \]
Remember, from vector field \( \mathbf{A}(\mathbf{r}) \), we can the magnitude and direction of the discrete vector \( \mathbf{A} \) that is located at the point defined by position vector \( \mathbf{r} \).

This discrete vector \( \mathbf{A} \) does not “extend” from the origin to the point described by position vector \( \mathbf{r} \). Rather, the discrete vector \( \mathbf{A} \) describes a quantity at that point, and that point only. The magnitude of vector \( \mathbf{A} \) does not have units of distance! The length of the arrow that represents vector \( \mathbf{A} \) is merely symbolic—its length has no direct physical meaning.

On the other hand, the position vector \( \mathbf{r} \), being a directed distance, does extend from the origin to a specific point in space. The magnitude of a position vector \( \mathbf{r} \) is distance—the length of the position vector arrow has a direct physical meaning!

Additionally, we should again note that a vector field need not be static. A dynamic vector field is likewise a function of time, and thus can be described with the notation:

\[
\mathbf{A}(\mathbf{r}, t)
\]