Voltage and Electric Potential

An important application of the line integral is the calculation of work. Say there is some vector field $\mathbf{F}(\mathbf{r})$ that exerts a force on some object.

**Q:** How much work ($W$) is done by this vector field if the object moves from point $P_a$ to $P_b$, along contour $C$?

**A:** We can find out by evaluating the line integral:

$$W_{ab} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{l}$$

Say this object is a charged particle with charge $Q$, and the force is applied by a static electric field $\mathbf{E}(\mathbf{r})$. We know the force on the charged particle is:

$$\mathbf{F}(\mathbf{r}) = Q \mathbf{E}(\mathbf{r})$$
and thus the work done by the electric field in moving a charged particle along some contour $C$ is:

$$W_{ab} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{l} = Q \int_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l}$$

Q: Oooh, I don’t like evaluating contour integrals; isn’t there some easier way?

A: Yes there is! Recall that a static electric field is a conservative vector field. Therefore, we can write any electric field as the gradient of a specific scalar field $V(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

We can then evaluate the work integral as:

$$W_{ab} = Q \int_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l}$$

$$= -Q \int_C \nabla V(\mathbf{r}) \cdot d\mathbf{l}$$

$$= -Q \left[ V(\mathbf{r}_b) - V(\mathbf{r}_a) \right]$$

$$= Q \left[ V(\mathbf{r}_a) - V(\mathbf{r}_b) \right]$$
We define:

\[ V_{ab} \doteq V(r_a) - V(r_b) \]

Therefore:

\[ W_{ab} = Q \cdot V_{ab} \]

Q: So what the heck is \( V_{ab} \)? Does it mean anything? Do we use it in engineering?

A: First, consider what \( W_{ab} \) is!

The value \( W_{ab} \) represents the work done by the electric field on charge \( Q \) when moving it from point \( P_a \) to point \( P_b \). This is precisely the same concept as when a gravitational force field moves an object from one point to another.

The work done by the gravitational field in this case is equal to the difference in potential energy (P.E.) between the object at these two points.
The value $W_{ab}$ represents the same thing! It is the difference in potential energy between the charge at point $P_a$ and at $P_b$.

**Q:** Great, now we know what $W_{ab}$ is. But the question was, **WHAT IS $V_{ab}$?!**

**A:** That’s easy! Just rearrange the above equation:

$$V_{ab} = \frac{W_{ab}}{Q}$$

See? The value $V_{ab}$ is equal to the difference in potential energy, per coulomb of charge!

* In other words $V_{ab}$ represents the difference in potential energy for each coulomb of charge in $Q$.

* Another way to look at it: $V_{ab}$ is the difference in potential energy if the particle has a charge of 1 Coulomb (i.e., $Q = 1$).

Note that $V_{ab}$ can be expressed as:

$$V_{ab} = \int_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l}$$

$$= V(\mathbf{r}_a) - V(\mathbf{r}_b)$$

where point $P_a$ lies at the beginning of contour $C$, and $P_b$ lies at the end.
We refer to the scalar field $V(\vec{r})$ as the electric potential function, or the electric potential field.

We likewise refer to the scalar value $V_{ab}$ as the electric potential difference, or simply the potential difference between point $P_a$ and point $P_b$.

Note that $V_{ab}$ (and therefore $V(\vec{r})$), has units of:

$$V_{ab} = \frac{W_{ab}}{Q} \left[ \frac{\text{Joules}}{\text{Coulomb}} \right]$$

Joules/Coulomb is a rather awkward unit, so we will use the other name for it—VOLTS!

$$\frac{1 \text{ Joule}}{\text{Coulomb}} \doteq 1 \text{ Volt}$$

Q: Hey! We used volts in circuits class. Is this the same thing?

A: It is precisely the same thing!

Perhaps this will help. Say $P_a$ and $P_b$ are two points somewhere on a circuit. But let’s call these points something different, say point + and point −.
Therefore, $V$ represents the potential difference (in volts) between point (i.e., node) + and point (node) -. Note this value can be either positive or negative.

Q: But, does this mean that circuits produce electric fields?

A: Absolutely! Anytime you can measure a voltage (i.e., a potential difference) between two points, an electric field must be present!