2.3 Vector Algebra (pp. 11-16)

A. Arithmetic Operations of Vectors

HO: Arithmetic Operations of Vectors

B. Arithmetic Operations of Vectors and Scalars

C. Multiplicative Operations of Vectors and Scalars

HO: Multiplicative Operations of Vectors and Scalars

HO: The Unit Vector

D. Multiplicative Operations of Vectors

HO: The Dot Product

HO: The Cross Product

HO: The Triple Product

E. Vectors Algebra

HO: Example: Vector Algebra

HO: Scalar, Vector, or Neither?

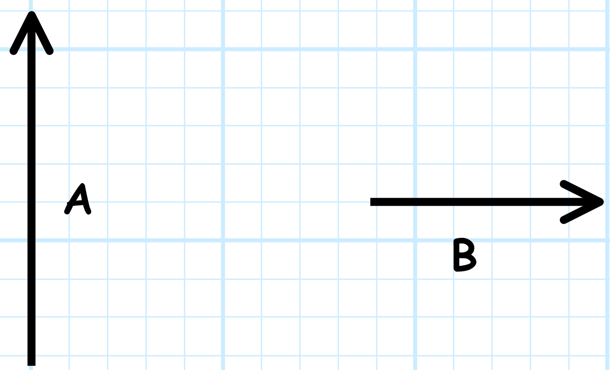
F. Orthogonal and Orthonormal Vector Sets

HO: Orthogonal and Orthonormal Vector Sets

Arithmetic Operations of Vectors

Vector Addition

Consider two vectors, denoted **A** and **B**.

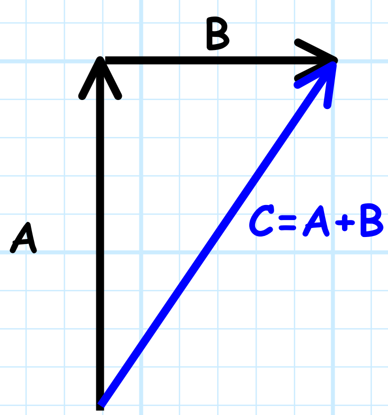


Q: *Say we **add** these two vectors together; what is the **result**?*

A: The addition of two vectors results in **another vector**, which we will denote as **C**. Therefore, we can say:

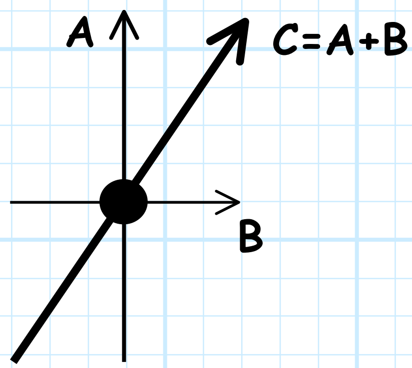
$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

The **magnitude** and **direction** of **C** is determined by the **head-to-tail rule**.



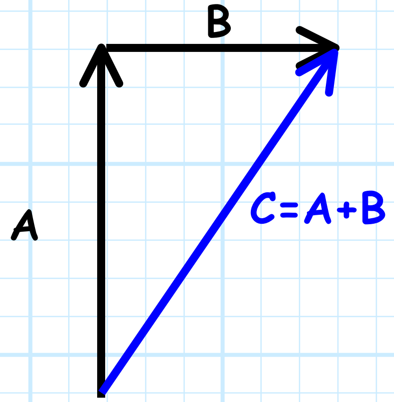
This is not a **provable** result, rather the head-to-tail rule is the **definition** of vector addition. This definition is used because it has many **applications** in physics.

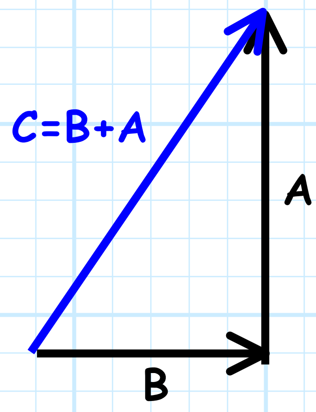
For **example**, if vectors **A** and **B** represent two **forces** acting an object, then vector **C** represents the **resultant force** when **A** and **B** are simultaneously applied.



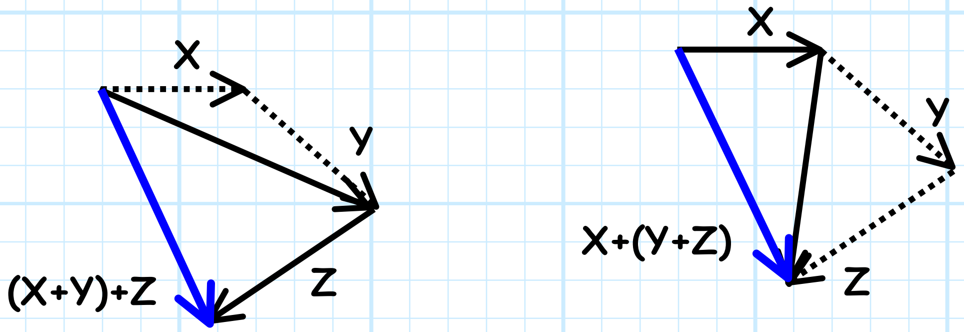
Some important properties of vector addition:

1. Vector addition is **commutative** - **A+B = B+A**

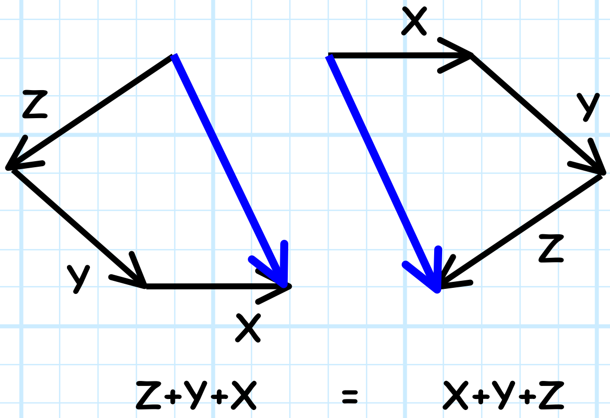




2. Vector addition is **associative** - (X+Y) + Z = X + (Y+Z)



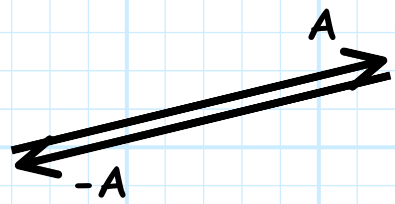
From these two properties, we can conclude that the addition of **several** vectors can be executed in **any order**:



Vector Subtraction

First, we define the **negative** of a vector to be a vector with **equal magnitude** but **opposite direction**.

Note that **A** + **(-A)** = 0



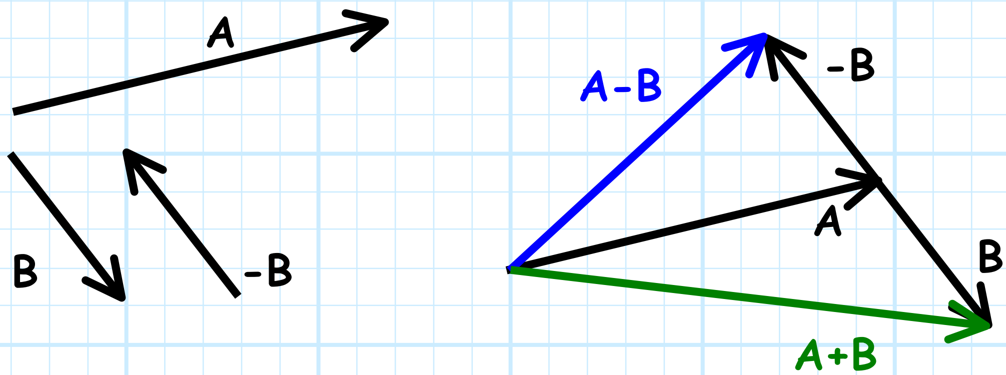
We can therefore consider the addition of a negative vector as a **subtraction**:

$$\mathbf{A} - \mathbf{A} = 0$$

More generally, we can write:

$$\boxed{\mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}}$$

E.G.,



Q: ***Is A-B = B-A ?***

A: What do **you** think ?

Multiplicative Operations of Vectors and Scalars

Consider a scalar quantity a and a vector quantity **B**. We express the multiplication of these two values as:

$$a\mathbf{B} = \mathbf{C}$$

In other words, the product of a scalar and a vector is a **vector**!

Q: *OK, but what **is** vector **C** ? What is the **meaning** of a **B** ?*

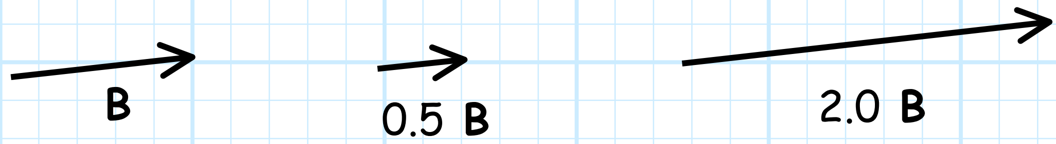
A: The resulting vector **C** has a **magnitude** that is equal to a times the **magnitude** of **B**. In other words:

$$|\mathbf{C}| = a\,|\mathbf{B}|$$

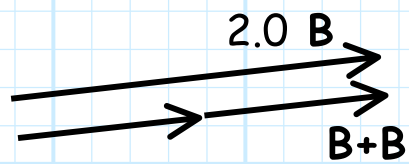
However, the **direction** of vector **C** is **exactly** that of **B**.

Therefore multiplying a vector by a scalar changes the **magnitude** of the vector, but **not** its direction.

For example:



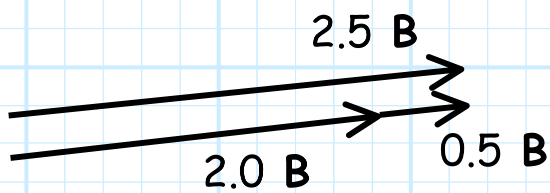
Note that **B** + **B** = 2.0 **B** !



1. More generally, we find that scalar-vector multiplication is **distributive**:

$$a\,\mathbf{B} + b\,\mathbf{B} = (a+b)\,\mathbf{B}$$

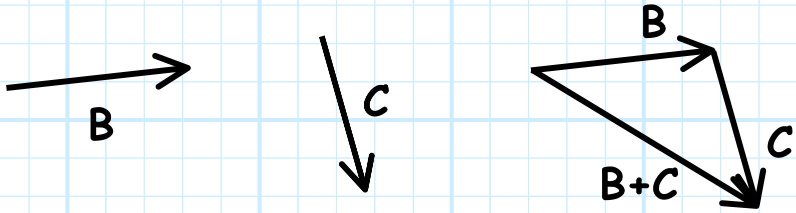
E.G.,

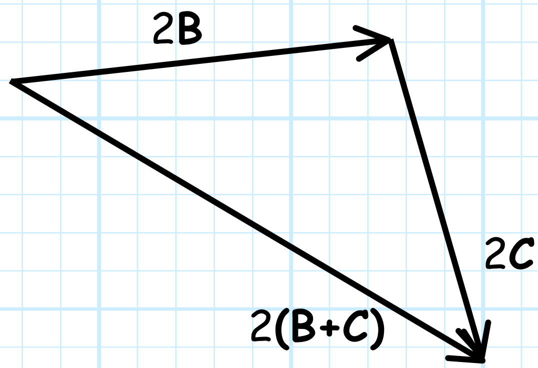


2. And **also distributive** as:

$$a\,\mathbf{B} + a\,\mathbf{C} = a\,(\mathbf{B} + \mathbf{C})$$

E.G.,





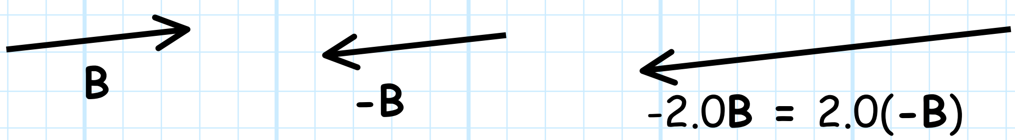
3. Scalar-Vector multiplication is also **commutative**:

$$a\,\mathbf{B} = \mathbf{B}\,a$$

4. Multiplication of a vector by a **negative** scalar is interpreted as:

$$-a\,\mathbf{B} = a\,(-\mathbf{B})$$

E.G.,



5. **Division** of a vector by a scalar is the same as multiplying the vector by the **inverse** of the scalar:

$$\frac{\mathbf{B}}{a} = \left(\frac{1}{a}\right)\mathbf{B}$$

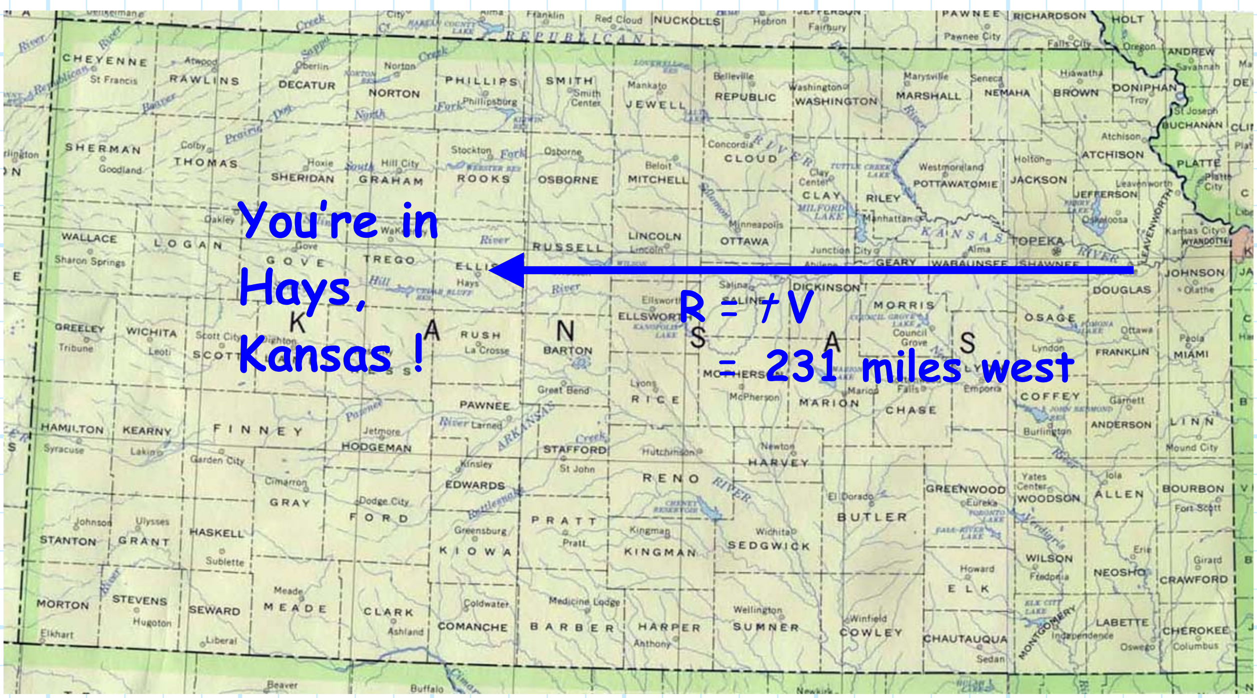
Scalar-Vector multiplication is likewise used in many **physical** applications. For example, say you start in Lawrence and head **west** at **70 mph** for exactly **3.3 hours**.

Note your velocity has both **direction** (west) and **magnitude** (70 mph) – it's a **vector**! Lets denote it as $\mathbf{V}$ = 70 mph west.

Likewise, your travel time is a **scalar**; lets denote it as t = 3.3 h.

Now, lets **multiply** the two together (i.e., $t\,\mathbf{V}$). The **magnitude** of the resulting vector is 70(3.3) = **231 miles**. The **direction** of the resulting vector is of course **unchanged**: west.

A vector describing a distance and a direction—a **directed distance**! We find that $t\,\mathbf{V} = \overline{\mathbf{R}}$, where $\overline{\mathbf{R}}$ identifies your **location** after 3.3 hours!



The Unit Vector

Now that we understand multiplication and division of a vector by a scalar, we can discuss a very important concept: **the unit vector**.

Lets begin with vector **A**. Say we **divide** this vector by its **magnitude** (a scalar value). We create a new vector, which we will denote as $\hat{a}_A$:

$$\hat{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|}$$

Q: *How is vector $\hat{a}_A$ related to vector **A** ?*

A: Since we divided **A** by a scalar value, the vector $\hat{a}_A$ has the **same direction** as vector **A**.

But, the **magnitude** of $\hat{a}_A$ is:

$$|\hat{a}_A| = \frac{|\mathbf{A}|}{|\mathbf{A}|} = 1$$

The vector $\hat{a}_A$ has a magnitude equal to **one** ! We call such a vector a **unit vector**.

A unit vector is essentially a **description of direction** only, as its magnitude is always **unit valued** (i.e., equal to one). Therefore:

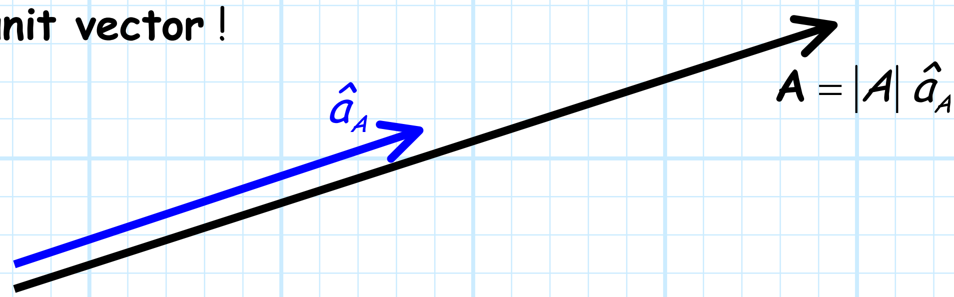
* $|\mathbf{A}|$ is a scalar value that describes the **magnitude** of vector **A**.

* $\hat{a}_A$ is a vector that describes the **direction** of **A**.

Rearranging the formula on the previous page, we can express vector **A** as:

$$\mathbf{A} = |A|\ \hat{a}_A$$

Thus, **any** vector can be written as the product of its **magnitude** and its **unit vector**!



Q: *What, **approximately**, is the **magnitude** of vector **A** shown above?*

A: $|\mathbf{A}| =$

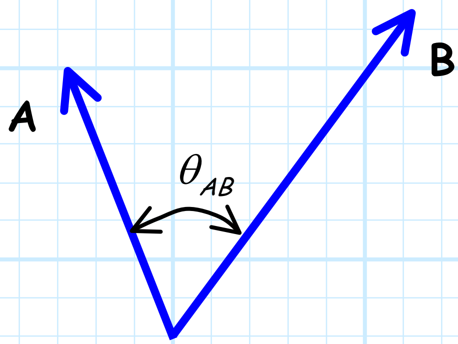
The Dot Product

The **dot product** of two vectors, A and B, is **denoted** as $\mathbf{A} \cdot \mathbf{B}$.

The dot product of two vectors is **defined** as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta_{AB}$$

where the angle θ_{AB} is the angle formed **between** the vectors **A** and **B**.



$$0 \leq \theta_{AB} \leq \pi$$

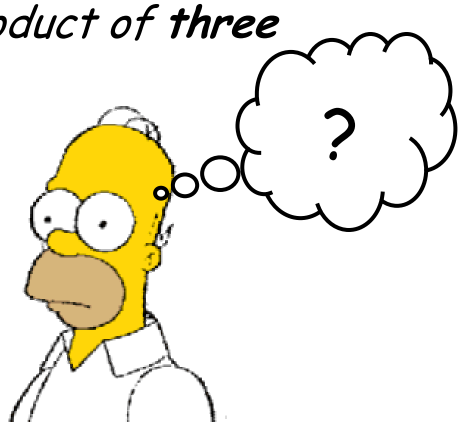
IMPORTANT NOTE: The dot product is an operation involving **two vectors**, but the result is a **scalar** !! E.G.,:

$$\mathbf{A} \cdot \mathbf{B} = c$$

The dot product is also called the **scalar product** of two vectors.

Q1: *So, what would the dot product of* ***three*** *vectors be? I.E.,:*

$$\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = ??$$



A:

Note also that the dot product is **commutative**, i.e.,:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

Some **more** fun facts about the dot product include:

1. The dot product of a vector **with itself** is equal to the **magnitude** of the vector **squared**.

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos 0^\circ = |\mathbf{A}|^2$$

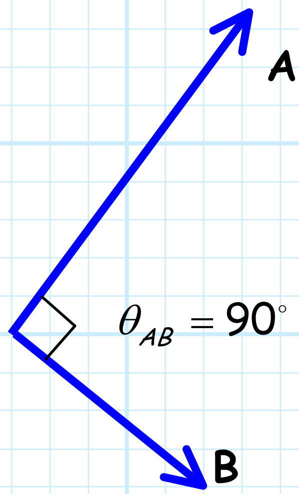
Therefore:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

2. If $\mathbf{A}\bullet\mathbf{B} = 0$ (and $|\mathbf{A}| \neq 0, |\mathbf{B}| \neq 0$), then it must be true that:

$$\cos\theta_{AB} = 0 \quad \Rightarrow \quad \theta_{AB} = 90^\circ$$

Thus, if $\mathbf{A}\bullet\mathbf{B} = 0$, the two vectors are **orthogonal** (perpendicular).

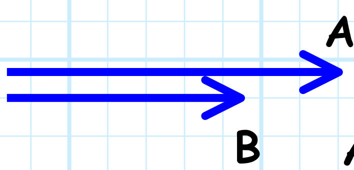


$$\mathbf{A}\bullet\mathbf{B} = 0$$

3. If $\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}||\mathbf{B}|$, then it must be true that:

$$\cos\theta_{AB} = 1 \quad \Rightarrow \quad \theta_{AB} = 0$$

Thus, vectors **A** and **B** must have the **same direction**. They are said to be **collinear** (parallel).



$$\begin{aligned}\mathbf{A}\bullet\mathbf{B} &= |\mathbf{A}||\mathbf{B}|\cos 0 \\ &= |\mathbf{A}||\mathbf{B}|\end{aligned}$$

4. If $\mathbf{A} \bullet \mathbf{B} = -|\mathbf{A}||\mathbf{B}|$, then it must be true that:

$$\cos\theta_{AB} = -1 \quad \Rightarrow \quad \theta_{AB} = 180^\circ$$

Thus, vectors **A** and **B** point in **opposite** directions; they are said to be **anti-parallel**.

$$\begin{aligned} \mathbf{A} \bullet \mathbf{B} &= |\mathbf{A}||\mathbf{B}| \cos 180^\circ \\ &= -|\mathbf{A}||\mathbf{B}| \end{aligned}$$

5. The dot product is **distributive** with addition, such that:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

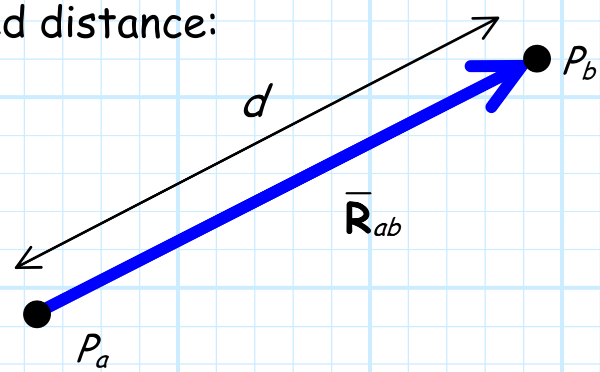
For example, we can write:

$$\begin{aligned} (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B}) \cdot \mathbf{A} + (\mathbf{A}+\mathbf{B}) \cdot \mathbf{C} && \text{(distributive)} \\ &= \mathbf{A} \cdot (\mathbf{A}+\mathbf{B}) + \mathbf{C} \cdot (\mathbf{A}+\mathbf{B}) && \text{(commutative)} \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B} && \text{(distributive)} \\ &= |\mathbf{A}|^2 + \mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B} \end{aligned}$$

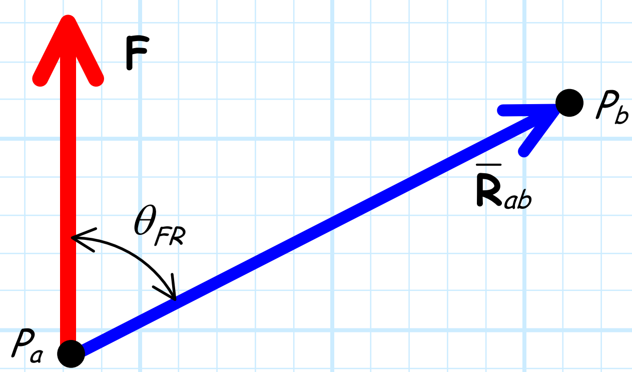
One application of the dot product is the determination of **work**. Say an object moves a distance d, directly from point P_a to point P_b, by applying a constant force **F**.

Q: How much work has been done?

First, we can specify the direct path from point P_a to point P_b with a directed distance:



The work done is simply the **dot product** of the applied force vector and the directed distance!



A:

$$\begin{aligned} W &= \mathbf{F} \cdot \bar{\mathbf{R}}_{ab} \\ &= |\mathbf{F}|\,|\bar{\mathbf{R}}_{ab}| \cos\theta_{FR} \\ &= d\,|\mathbf{F}| \cos\theta_{FR} \end{aligned}$$

The value $|\mathbf{F}|\cos\theta_{FR}$ is said to be the **scalar component** of force **F** in the **direction** of directed distance $\bar{\mathbf{R}}_{ab}$

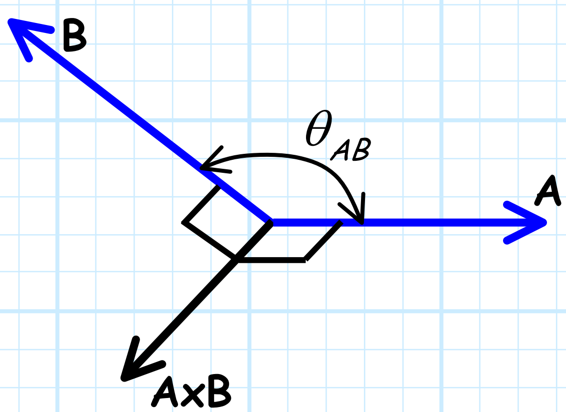
The Cross Product

The **cross product** of two vectors, **A** and **B**, is **denoted** as $\mathbf{A}\times\mathbf{B}$.

The cross product of two vectors is **defined** as:

$$\mathbf{A}\times\mathbf{B} = \hat{a}_n\,|\mathbf{A}||\mathbf{B}|\sin\theta_{AB}$$

Just as with the dot product, the angle θ_{AB} is the angle between the vectors **A** and **B**. The unit vector $\hat{a}_n$ is **orthogonal** to both **A** and **B** (i.e., $\hat{a}_n\cdot\mathbf{A}=0$ and $\hat{a}_n\cdot\mathbf{B}=0$).



$$0 \le \theta_{AB} \le \pi$$

IMPORTANT NOTE: The cross product is an operation involving **two vectors**, and the result is also a **vector**. E.G.,:

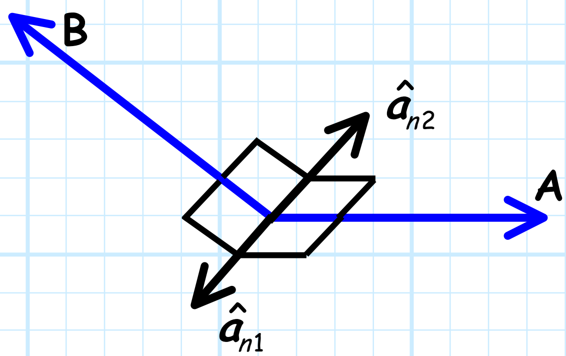
$$\mathbf{A}\times\mathbf{B} = \mathbf{C}$$

The **magnitude** of vector **AxB** is therefore:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta_{AB}$$

Whereas the **direction** of vector **AxB** is described by unit vector $\hat{a}_n$.

Problem! There are **two** unit vectors that satisfy the equations $\hat{a}_n \cdot \mathbf{A} = 0$ and $\hat{a}_n \cdot \mathbf{B} = 0$!! These two vectors are **anti-parallel**.



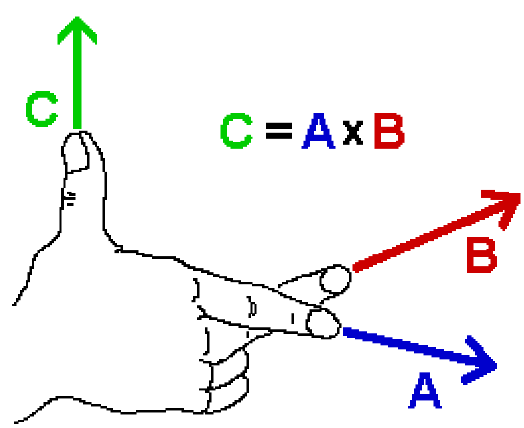
$$\mathbf{A} \cdot \hat{a}_{n1} = \mathbf{A} \cdot \hat{a}_{n2} = 0$$

$$\mathbf{B} \cdot \hat{a}_{n1} = \mathbf{B} \cdot \hat{a}_{n2} = 0$$

$$\hat{a}_{n1} = -\hat{a}_{n2}$$

Q: *Which unit vector is correct?*

A: Use the **right-hand rule** (See figure 2-9)!!

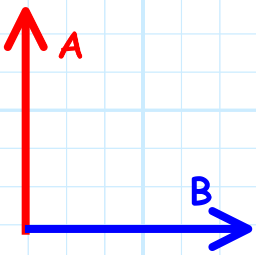


From: www.physics.udel.edu/~watson/phys345/class/1-right-hand-rule.html

Some fun facts about the **cross product** !

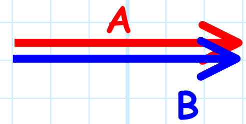
1. If $\theta_{AB} = 90^\circ$ (i.e., **orthogonal**), then:

$$\mathbf{A} \times \mathbf{B} = \hat{a}_n |\mathbf{A}||\mathbf{B}| \sin 90^\circ$$
$$= \hat{a}_n |\mathbf{A}||\mathbf{B}|$$

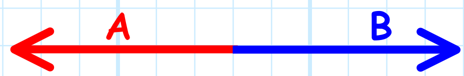


2. If $\theta_{AB} = 0^\circ$ (i.e., **parallel**), then:

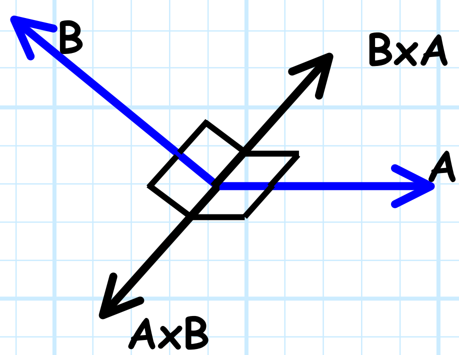
$$\mathbf{A} \times \mathbf{B} = \hat{a}_n |\mathbf{A}||\mathbf{B}| \sin 0^\circ$$
$$= 0$$

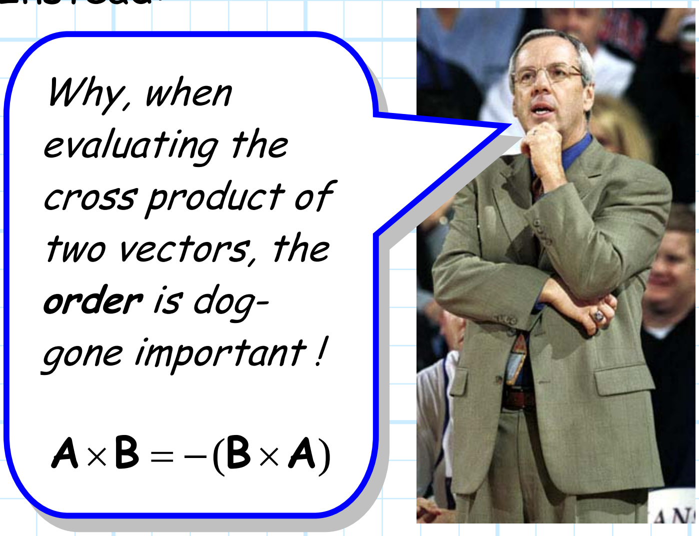


Note that **AxB** =0 also if $\theta_{AB} = 180^\circ$



3. The cross product is **not** commutative ! In other words, $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Instead:





4. The **negative** of the cross product is:

$$-(\mathbf{A} \times \mathbf{B}) = -\mathbf{A} \times \mathbf{B} = \mathbf{A} \times (-\mathbf{B})$$

5. The cross product is also **not** associative:

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\neq\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

Therefore, **AxBxC** has **ambiguous** meaning !

6. But, the cross product is **distributive**, in that:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$$

and also,

$$(\mathbf{B}+\mathbf{C})\times\mathbf{A}=(\mathbf{B}\times\mathbf{A})+(\mathbf{C}\times\mathbf{A})$$

The Triple Product

The **triple product** is not a "new" operation, as it is simply a combination of the **dot** and **cross** products.

The triple product of vectors **A**, **B**, and **C** is **denoted** as:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$$

Q: *Yikes! Does this mean:*

$$(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$$

or

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

A: The answer is **easy** ! Only one of these two interpretations makes sense:

In the **first** case, $\mathbf{A}\cdot\mathbf{B}$ is a scalar value, say $d=\mathbf{A}\cdot\mathbf{B}$. Therefore we can write the first equation as:

$$(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}=d\times\mathbf{C}$$

But, this makes no sense! The cross product of a **scalar** and a vector has **no meaning**.

In the **second** interpretation, the cross product $\mathbf{B}\times\mathbf{C}$ is a **vector**, say $\mathbf{B}\times\mathbf{C}=\mathbf{D}$. Therefore, we can write the second equation as:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{A}\cdot\mathbf{D}$$

Not only does this make sense, but the result is a **scalar** !

> The triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ results in a **scalar value**.

<u>The Cyclic Property</u>

It can be shown that the triple product of vectors **A, B,** and **C** can be evaluated in three ways:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \quad = \quad \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} \quad = \quad \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$$

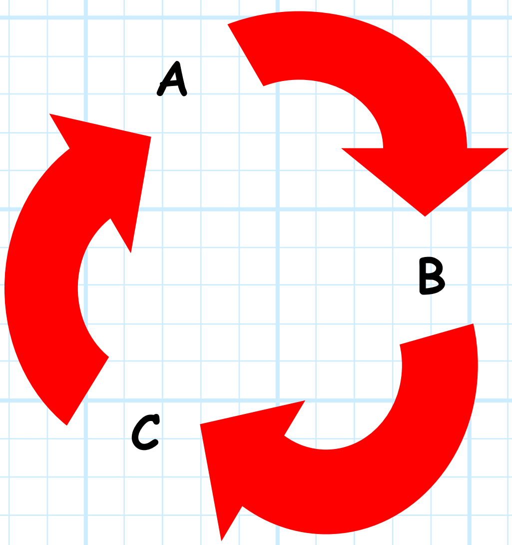
But, it is important to note that this does **not** mean that order is unimportant! For example:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{A}\cdot\mathbf{C}\times\mathbf{B}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{C}\cdot\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{B}\cdot\mathbf{A}\times\mathbf{C}$$

The **cyclical** rule means that the triple product is **invariant** to **shifts** (i.e., rotations) in the order of the vectors.



There are **six ways** to arrange three vectors. Therefore, we can group the triple product of three vectors into **two groups** of **three products**:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$$

$$\mathbf{B}\cdot\mathbf{A}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{B}\times\mathbf{A} = \mathbf{A}\cdot\mathbf{C}\times\mathbf{B}$$

but, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = -\left(\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}\right)$

Example: Vector Algebra

Consider the **scalar** expression:

$$ac + bc + bd + ad$$

We can manipulate and simply this expression using the rules of **scalar algebra**:

$$\begin{aligned} ac + bc + bd + ad &= ac + ad + bc + bd && \text{(commutative)} \\ &= (ac + ad) + (bc + bd) && \text{(associative)} \\ &= a(c + d) + b(c + d) && \text{(distributive)} \\ &= (a + b)(c + d) && \text{(distributive)} \end{aligned}$$

We can likewise perform a similar analysis on vector expressions! Consider now the expression:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} \times \mathbf{A}$$

We can show that this is actually a very familiar and basic vector operation!

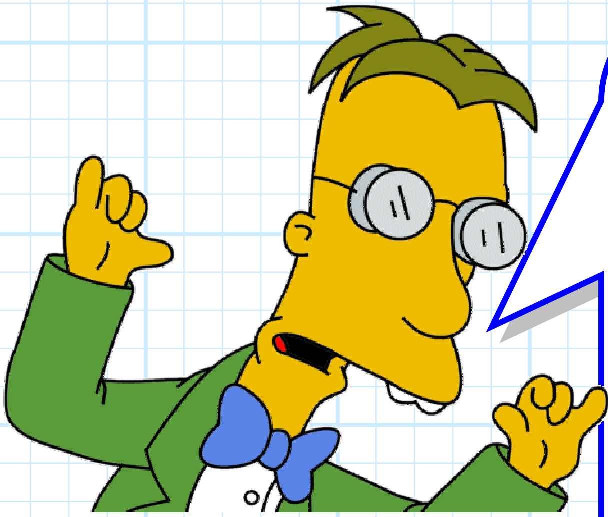
$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} \times \mathbf{A} &= \mathbf{C} \cdot \mathbf{A} \times (\mathbf{A} + \mathbf{B}) && \text{(Triple product identity)} \\ &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}) && \text{(Cross Product Distibutive)} \\ &= \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} && (\text{Since } \mathbf{A} \times \mathbf{A} = 0) \\ &= \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} && \text{(Triple product identity)} \end{aligned}$$

Or, for example, if we consider:

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+2\mathbf{A})$$

we find:

$$
\begin{aligned}
&=(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+2\mathbf{A}) && \\
&=\mathbf{A}\cdot(\mathbf{B}+2\mathbf{A})+\mathbf{B}\cdot(\mathbf{B}+2\mathbf{A}) && \text{(dot product distributive)}\\
&=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot 2\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+\mathbf{B}\cdot 2\mathbf{A} && \text{(dot product distributive)}\\
&=\mathbf{A}\cdot\mathbf{B}+2\,\mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+2\,\mathbf{B}\cdot\mathbf{A} && \text{(scalar multiply commutative)}\\
&=\mathbf{A}\cdot\mathbf{B}+2|\mathbf{A}|^2+|\mathbf{B}|^2+2\,\mathbf{B}\cdot\mathbf{A} && (\mathbf{C}\cdot\mathbf{C}=|\mathbf{C}|^2 \text{ identity})\\
&=\mathbf{A}\cdot\mathbf{B}+2|\mathbf{A}|^2+|\mathbf{B}|^2+2\,\mathbf{A}\cdot\mathbf{B} && \text{(dot product communitive)}\\
&=2|\mathbf{A}|^2+2\,\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && \text{(vector addition commutative)}\\
&=2|\mathbf{A}|^2+(2+1)\,\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && \text{(scalar multiply distributive)}\\
&=|\mathbf{A}|^2+3\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && (2+1=3)
\end{aligned}
$$



Keep in mind one very important point when doing vector algebra—the expression ***can never change type*** *(e.g., from vector to scalar)!*

In other words, if the expression ***initially*** *results in a vector (or scalar), then after each manipulation, the result must* ***also*** *be a vector (or scalar).*

For example, we find that the following expression **cannot** possibly be true!

$$\mathbf{A} \times \mathbf{B} + (\mathbf{B} \cdot \mathbf{C})\mathbf{A} = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) + (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$$

Q: *Do **you** see why?*

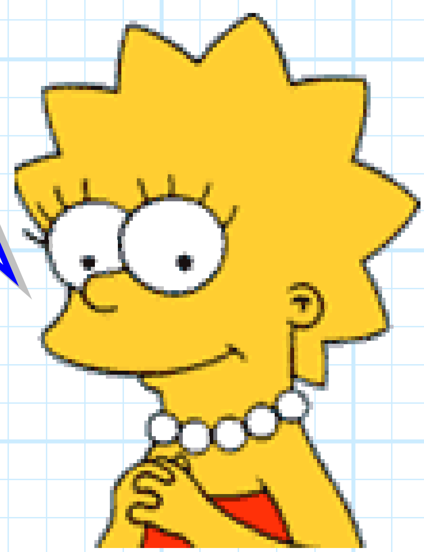
A: ____

*Likewise, be careful **not** to create expressions that have **no** mathematical meaning whatsoever!*

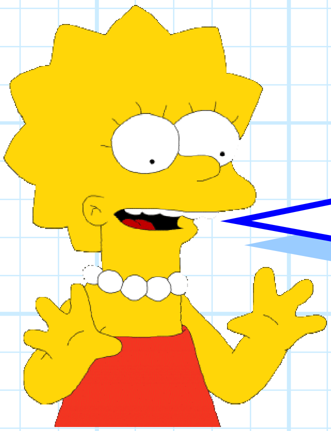
Examples include:

$$(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$$

$$\mathbf{A} + (\mathbf{B} \cdot \mathbf{C})$$



Scalar, Vector, or Neither?



*Let's **test** our vector algebraic skills! Can **you** evaluate the following expressions, and determine whether the result is a **scalar** (S), a **vector** (V), or **neither** (N) ??*

1. $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ ________
2. $\mathbf{A} + (\mathbf{B} \cdot \mathbf{C})$ ________
3. $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$ ________
4. $\mathbf{A}(\mathbf{B} \times \mathbf{C})$ ________
5. $\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ ________
6. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \cdot (\mathbf{A} + \mathbf{B})$ ________
7. $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \cdot \mathbf{D}$ ________

Orthogonal and Orthonormal Vector Sets

We often specify or relate a set of **scalar values** (e.g., x, y, z) using a set of **scalar equations**. For example, we might say:

$$x = y \quad \text{and} \quad z = x + 2$$

From which we can conclude a **third** expression:

$$z = y + 2$$

Say that we now add a **new** constraint to the first two:

$$x + y = 2$$

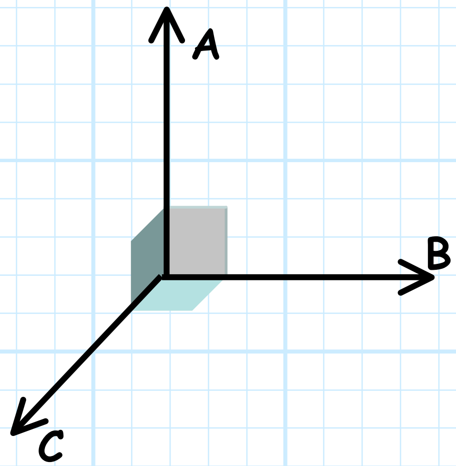
We can now **specifically** conclude that:

$$x = 1 \qquad y = 1 \qquad z = 3$$

Note we can likewise use **vector equations** to specify or relate a set of **vectors** (e.g., $\mathbf{A}, \mathbf{B}, \mathbf{C}$).

For example, consider a set of **three** vectors that are oriented such that they are **mutually orthogonal** !

In other words, each vector is **perpendicular** to each of the other two:



Note that we can **describe** this orthogonal relationship mathematically using **three simple equations**:

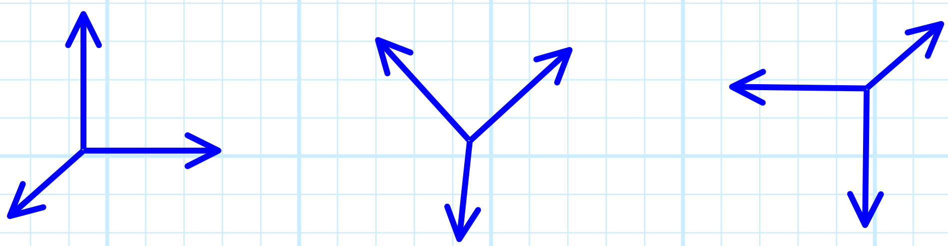
$$\mathbf{A} \cdot \mathbf{B} = 0$$
$$\mathbf{A} \cdot \mathbf{C} = 0$$
$$\mathbf{B} \cdot \mathbf{C} = 0$$

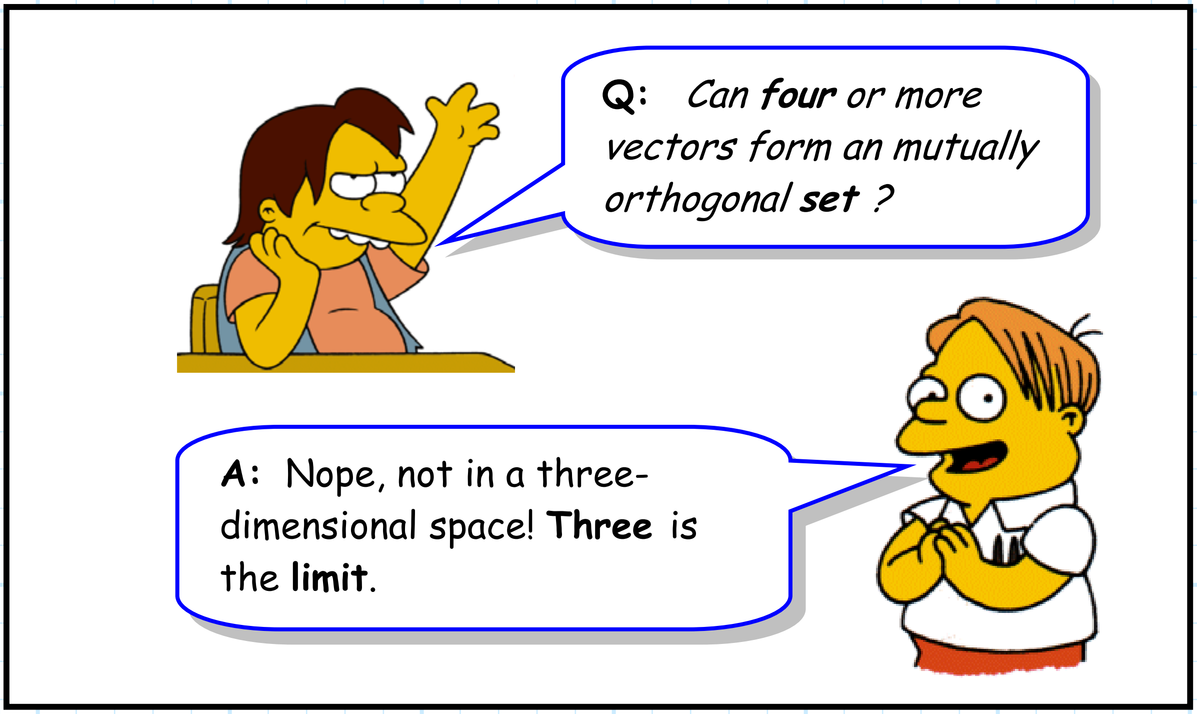
We can therefore **define** an orthogonal set of vectors using the **dot product**:

Three (non-zero) vectors **A**, **B** and **C** form an **orthogonal set** iff they satisfy $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = 0$

Note that there are an **infinite** number of mutually orthogonal vector **sets** that can be formed !

E.G:





Consider now a mutually orthogonal set of **unit vectors**. Such a set can be defined as three vectors that satisfy these **six** equations:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = 0 \qquad \text{(mutually orthognal)}$$

$$\mathbf{A} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{C} = 1 \qquad \text{(unit magnitudes)}$$

A set of vectors that satisfy these equations are said to form an **orthonormal** set of vectors ! Therefore, an orthonormal set consists of **unit vectors** where:

$$\hat{a}_A \cdot \hat{a}_B = \hat{a}_B \cdot \hat{a}_C = \hat{a}_C \cdot \hat{a}_A = 0$$

Again, there are an **infinite** number of **orthonormal** vector sets, but each set consists of only **three** vectors.