3-5 Two Action-at-a-Distance Laws

Reading Assignment: *pp. 71-75*

A. Coulomb's Law of Force

Q:

A:

HO: Coulomb's Law

HO: The Vector Form of Coulomb's Law

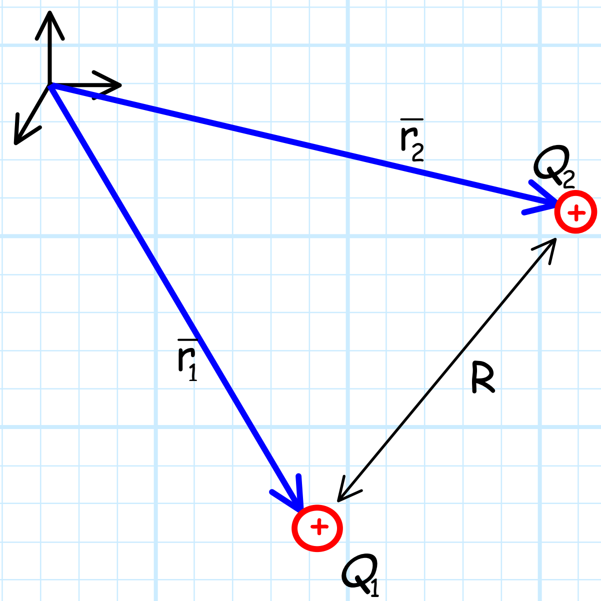
B. Ampere's Law of Force

HO: Ampere's Law of Force

Example: Ampere's Law of Force

Coulomb's Law of Force

Consider **two** point charges, Q_1 and Q_2, located at positions $\bar{r}_1$ and $\bar{r}_2$, respectively.



We will find that **each** charge has a **force F** (with magnitude and direction) exerted on it.

This force is **dependent** on both the **sign** (+ or -) and the **magnitude** of charges Q_1 and Q_2, as well as the **distance** R between the charges.

Charles Coulomb determined this relationship in the 18th century! We call his result **Coulomb's Law**:

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\,\hat{a}_{21} \quad [N]$$

This force $\mathbf{F}_1$ is the force exerted **on** charge Q_1. Likewise, the force exerted **on** charge Q_2 is equal to:

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_2\,Q_1}{R^2}\,\hat{a}_{12} \quad [N]$$

In these formula, the value ε_0 is a **constant** that describes the **permittivity of free space** (i.e., a vacuum).

$$\varepsilon_0 \doteq \text{permittivity of free space}$$
$$= 8.854 \times 10^{-12} \quad \left[\frac{C^2}{Nm^2} = \frac{farads}{m}\right]$$

Note the **only difference** between the equations for forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are the **unit vectors** $\hat{a}_{21}$ and $\hat{a}_{12}$.

* Unit vector $\hat{a}_{21}$ points **from** the location of Q_2 (i.e., $\bar{r}_2$) **to** the location of charge Q_1 (i.e., $\bar{r}_1$).

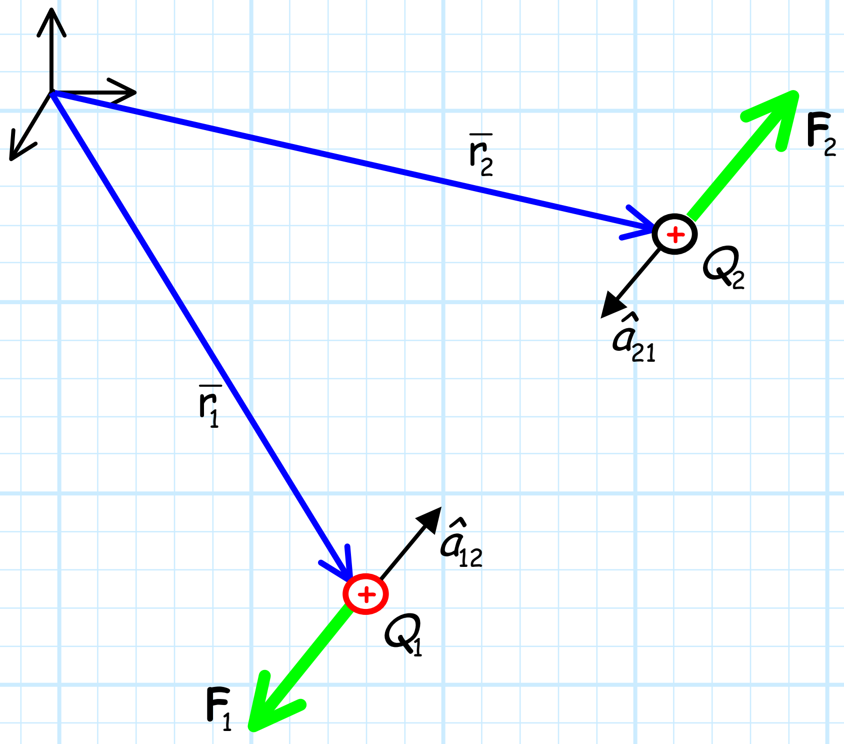
* Likewise, unit vector $\hat{a}_{12}$ points **from** the location of Q_1 (i.e., $\bar{r}_1$) **to** the location of charge Q_2 (i.e., $\bar{r}_2$).

Note therefore, that these unit vectors point in **opposite** directions, a result we express mathematically as $\hat{a}_{21} = -\hat{a}_{12}$.

Therefore we find:

$$\begin{aligned}\mathbf{F}_1 &= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\ \hat{a}_{21} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\ \left(-\hat{a}_{12}\right) \\ &= -\left(\frac{1}{4\pi\varepsilon_0}\frac{Q_2 Q_1}{R^2}\ \hat{a}_{12}\right) \\ &= -\mathbf{F}_2\end{aligned}$$

Look! Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have **equal magnitude**, but point in **opposite directions** !

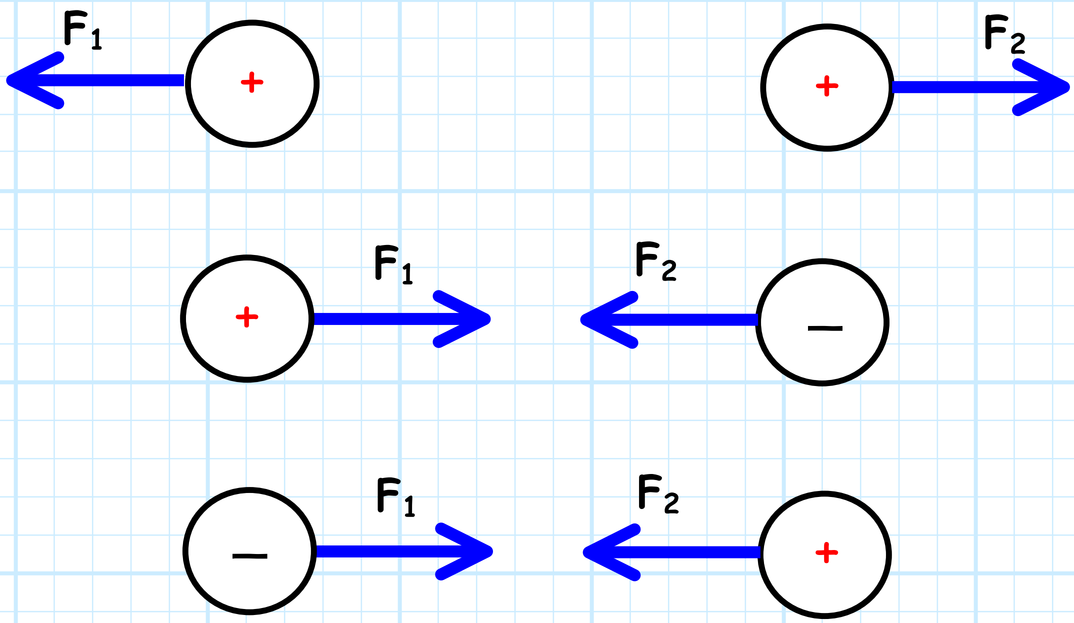


Note in the case shown above, **both** charges were **positive**.

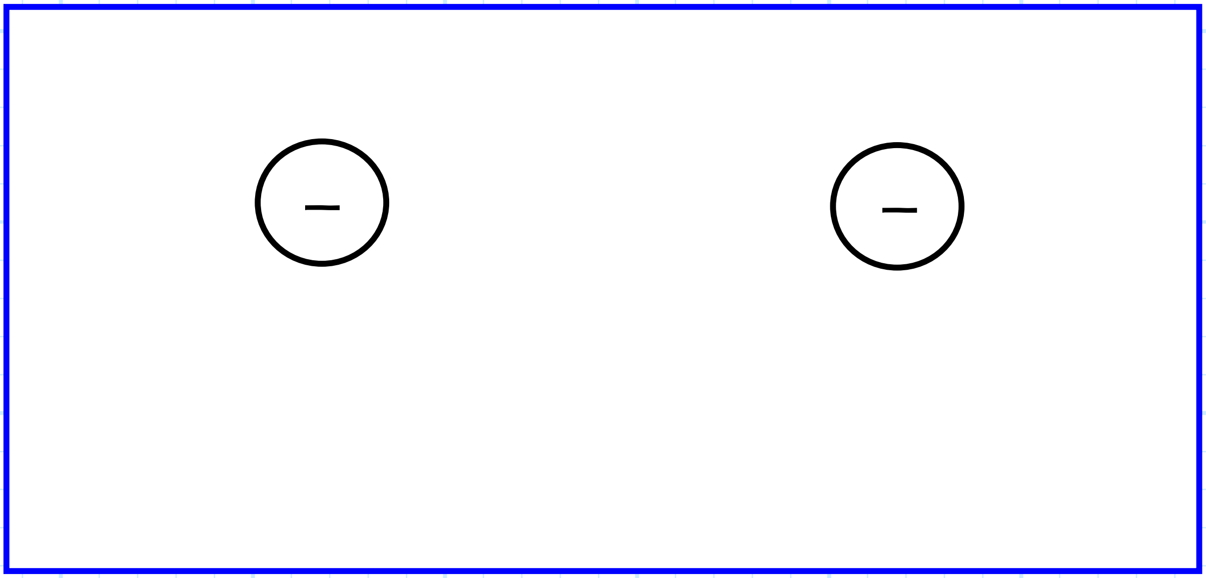
Q: *What happens when **one** of the charges is **negative**?*

A: Look at Coulomb's Law ! If one charge is positive, and the other is negative, then the **product** $Q_1 Q_2$ is **negative**. The resulting force vectors are therefore negative—they point in the **opposite** direction of the previous (i.e., both positive) case!

Therefore, we find that:



What about **this** case ?



We come to the **important** conclusion that:

1) charges of **opposite** sign **attract**.

2) charges with the **same** sign **repel**.

Charles-Augustin de Coulomb (1736-1806), a military civil engineer, retired from the French army because of ill health after years in the West Indies. Forced from Paris by the disturbances of the revolution, he began working at his family estate and discovered that the torsion characteristics of long fibers made them ideal for the sensitive measurement of **magnetic** and **electric** forces. He was familiar with Newton's **inverse-square law** and in the period 1785-1791 he succeeded in showing that **electrostatic** forces obey the **same** rule. (from *www.ee.umd.edu/~taylor/frame1.htm*)

The Vector Form of Coulomb's Law of Force

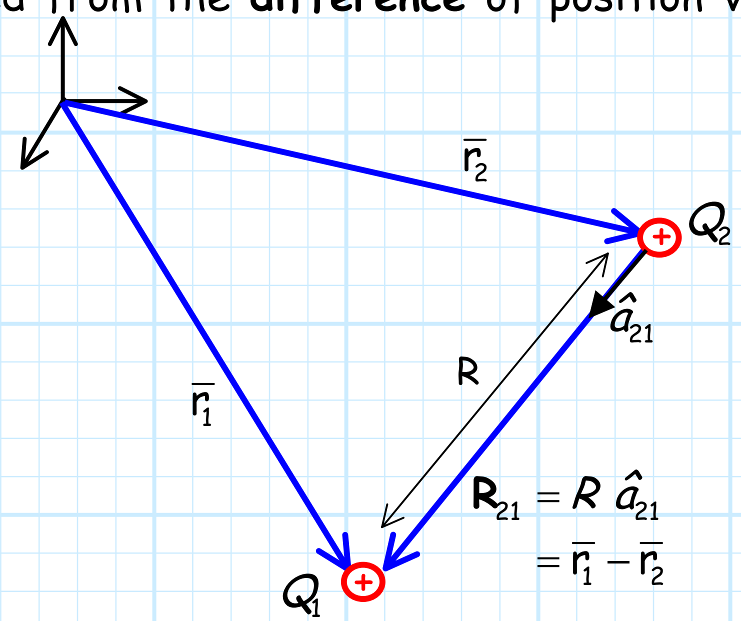
The **position vector** can be used to make the **calculations** of Coulomb's Law of Force more **explicit**. Recall:

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\ \hat{a}_{21} \quad [N]$$

Specifically, we ask ourselves the question: **how** do we determine the **unit vector** $\hat{a}_{21}$ and **distance** R??

* Recall the **unit** vector $\hat{a}_{21}$ is a unit vector directed **from** Q_2 **toward** Q_1, and R is the **distance** between the two charges.

* The **directed distance** vector $\mathbf{R}_{21} = R\ \hat{a}_{21}$ can be determined from the **difference** of position vectors $\bar{r}_1$ and $\bar{r}_2$.



This directed distance $\mathbf{R}_{21} = \bar{r}_1 - \bar{r}_2$ is **all** we need to determine **both** unit vector $\hat{a}_{21}$ and distance R (i.e., $\mathbf{R}_{21} = R\,\hat{a}_{21}$)!

For example, since the **direction** of directed distance $\mathbf{R}_{21}$ is equal to $\hat{a}_{21}$, we can **explicitly** find this unit vector by **dividing** $\mathbf{R}_{21}$ by its **magnitude**:

$$\hat{a}_{21} = \frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|} = \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|}$$

Likewise, the **distance** R between the two charges is simply the magnitude of directed distance $\mathbf{R}_{21}$!

$$R = |\mathbf{R}_{21}| = |\bar{r}_1 - \bar{r}_2|$$

Using these expressions, we find that we can express **Coulomb's Law** entirely in terms of $\mathbf{R}_{21}$, the **directed distance** relating the location of Q_1 with respect to Q_2:

$$\begin{aligned}\mathbf{F}_1 &= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\,\hat{a}_{21} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{|\mathbf{R}_{21}|^2}\frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|} \\ &= \frac{Q_1 Q_2}{4\pi\varepsilon_0}\frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|^3}\end{aligned}$$

Explicitly using the relation $\mathbf{R}_{21} = \bar{r}_1 - \bar{r}_2$, we find:

$$\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|^3} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|^3}$$

We of course could likewise define a directed distance:

$$\mathbf{R}_{12} = \bar{r}_2 - \bar{r}_1$$

which relates the location of Q_2 with respect to Q_1.

We can thus describe the force on charge Q_2 as:

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|^3} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|^3}$$

Note since $\mathbf{R}_{12} = -\mathbf{R}_{21}$ (thus $|\mathbf{R}_{12}| = |\mathbf{R}_{21}|$), we again find that:

$$\mathbf{F}_2 = -\mathbf{F}_1$$

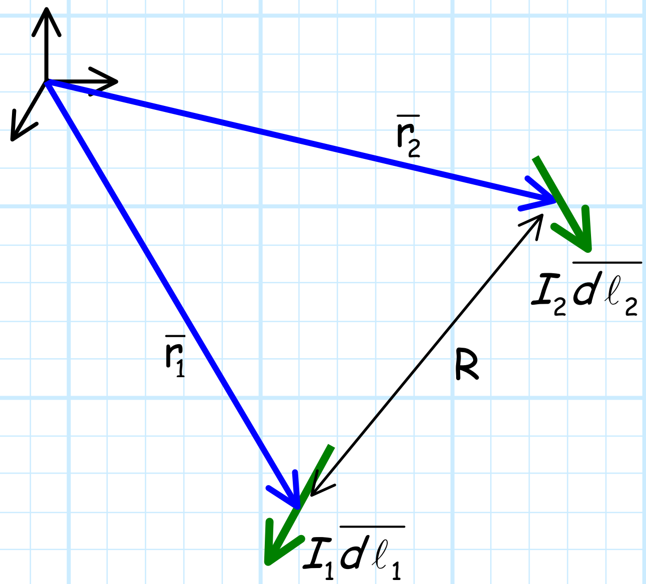
The forces on each charge have **equal** magnitude but **opposite** direction.

See Example 3-3 on pages 72-73 !

Ampere's Law of Force

Consider the case of two **current filaments** located in space.

One filament has current I_1 flowing along differential displacement distance $\overline{d\ell_1}$, while the **other** has current I_2 flowing along $\overline{d\ell_2}$.



We find that each current filament exerts **force** $d\mathbf{F}$ on the other!

The force depends on the **magnitude** and **direction** of **each** filament vector ($I\,\overline{d\ell}$), as well as on the **distance** R between the two currents.

Andre Ampere determined this relationship in the 18th century, and we call his result **Ampere's Law of Force**:

$$d\mathbf{F}_1 = \frac{\mu_0}{4\pi}\frac{I_1\overline{d\ell_1}\times\left(I_2\overline{d\ell_2}\times\hat{a}_{21}\right)}{R^2}$$

Q: *Yikes! What the heck does this mean ?*

A: Well,:

* The **unit vector** $\hat{a}_{21}$ is the unit vector directed from filament 2 to filament 1 (just like Coulomb's Law).

* The constant μ_0 is the **permeability of free space**, given as:

$$\mu_0 = 4\pi \times 10^{-7} \quad \left[N/A^2 = \frac{Henry}{meter} \right]$$

* The force $d\mathbf{F}_1$ is the force exerted **on** filament 1 **by** filament 2.

Q: *O.K., but what about:*

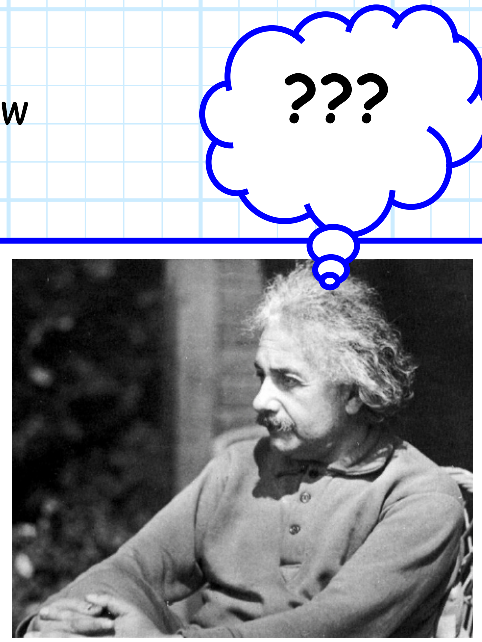
$$I_1 \overline{d\ell_1} \times \left(I_2 \overline{d\ell_2} \times \hat{a}_{21} \right) \quad ?!?$$

A: Using equation B.2 of your book (p. 639), we can rewrite this in terms of the **dot product**!

$$I_1\overline{d\ell_1}\times\left(I_2\overline{d\ell_2}\times\hat{a}_{21}\right)=\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\overline{d\ell_2}-\left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\hat{a}_{21}$$

Therefore, we can **also** write Ampere's Law of Force as:

$$d\mathbf{F}_1=\frac{\mu_0 I_1 I_2}{4\pi}\frac{\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\overline{d\ell_2}-\left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\hat{a}_{21}}{R^2}$$



*See! Didn't **that** help?*

Perhaps **not**. To interpret the result above, we need to look at several **examples**.

But first, let's examine one **very important** property of Ampere's Law of Force. Consider the force **on** filament 2 **by** filament 1—exactly the **opposite** case considered earlier.

We find from Ampere's Law of force:

$$d\mathbf{F}_2=\frac{\mu_0 I_2 I_1}{4\pi}\frac{\left(\overline{d\ell_2}\cdot\hat{a}_{12}\right)\overline{d\ell_1}-\left(\overline{d\ell_2}\cdot\overline{d\ell_1}\right)\hat{a}_{12}}{R^2}$$

Note in the **numerator** there are **two** vector terms. Let's **compare** them.

We find that the **second** terms in each force expression have **equal** magnitude but **opposite** direction, because $\hat{a}_{12} = -\hat{a}_{21}$.

$$\left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\hat{a}_{21} = -\left(\overline{d\ell_2}\cdot\overline{d\ell_1}\right)\hat{a}_{12}$$

However, the **first** vector terms in each expression are related in **neither** magnitude **nor** direction !

$$\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\overline{d\ell_2} \neq \left(\overline{d\ell_2}\cdot\hat{a}_{12}\right)\overline{d\ell_1}$$

Therefore, we discover that, in general, the force $d\mathbf{F}_1$ on filament 1, and the force $d\mathbf{F}_2$ on filament 2 are **not** related in **either** magnitude or in direction:

$$d\mathbf{F}_1 \neq d\mathbf{F}_2$$

In fact, we can have situations where the force on one element is **zero**, while the force on the other element is **not!**

This, of course, is much **different** than **Coulomb's Law of Force**, where we found that $\mathbf{F}_1 = -\mathbf{F}_2$ **always**.

André-Marie Ampère (1775-1836) was a child prodigy whose early life was marred by tragedy: Ampère's father was beheaded in his presence during the Revolution and, later, his wife died four years after their marriage. As a scientist, Ampère had flashes of inspiration which he would pursue to their conclusion. When he learned of Ørsted's discovery in 1820 that a **magnetic** needle is deflected by a varying nearby **current**, he prepared within a week the first of several papers on the theory of this phenomenon, formulating the law of **electromagnetism** (Ampère's law) that describes mathematically the **magnetic force** between two **circuits**. (*from www.ee.umd.edu/~taylor/frame3.htm*)

Example: Ampere's Law of Force

Let's again consider Ampere's Law of Force in the following form:

$$d\mathbf{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \frac{\left(\overline{d\ell_1} \cdot \hat{a}_{21}\right)\overline{d\ell_2} - \left(\overline{d\ell_1} \cdot \overline{d\ell_2}\right)\hat{a}_{21}}{R^2}$$

$$= \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \hat{a}_{21}\right)\overline{d\ell_2} \; - \; \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \overline{d\ell_2}\right)\hat{a}_{21}$$

It is apparent that we can consider the force on **filament 1** to consist of **two** forces, i.e.:

$$d\mathbf{F}_1 = d\mathbf{F}_1^a + d\mathbf{F}_1^b$$

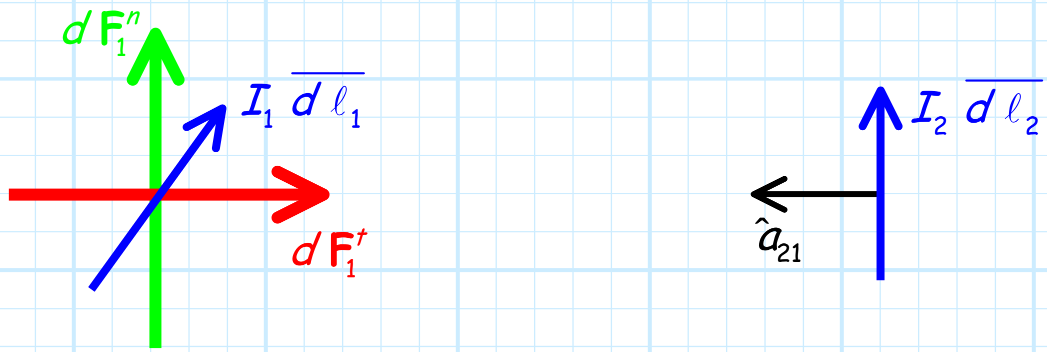
where

$$d\mathbf{F}_1^a = \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \hat{a}_{21}\right)\; \overline{d\ell_2}$$

and

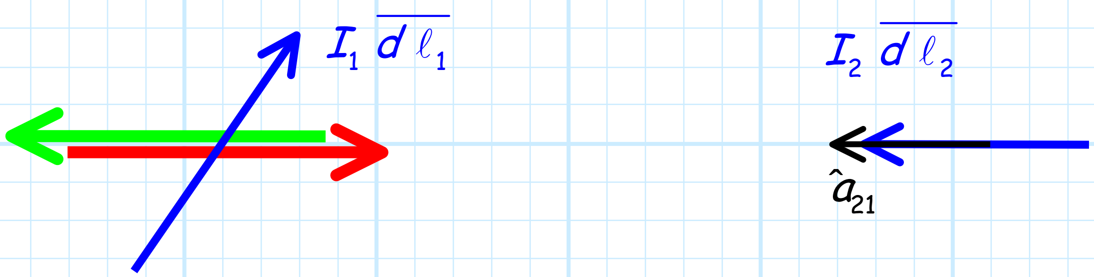
$$d\mathbf{F}_1^b = -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \overline{d\ell_2}\right)\; \hat{a}_{21}$$

Therefore, the force on filament **1** has a component in the direction $\overline{d\ell_2}$ (i.e., in the direction of current filament **2**), **and** a component in the direction $-\hat{a}_{21}$.



So, let's consider several **examples**:

Example 1: Filament **2** points **toward** filament **1**



Therefore, since $\overline{d\ell_2} = \left|\overline{d\ell_2}\right|\hat{a}_{21}$:

$$\begin{aligned}\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\overline{d\ell_2} &= \left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\left|\overline{d\ell_2}\right|\hat{a}_{21}\\ &= \left(\overline{d\ell_1}\cdot\left|\overline{d\ell_2}\right|\hat{a}_{21}\right)\hat{a}_{21}\\ &= \left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\hat{a}_{21}\end{aligned}$$

and therefore:

$$d\mathbf{F}_1^a = -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\overline{d\ell_2}$$

$$= -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\hat{a}_{21}$$

$$= -d\mathbf{F}_1^b$$

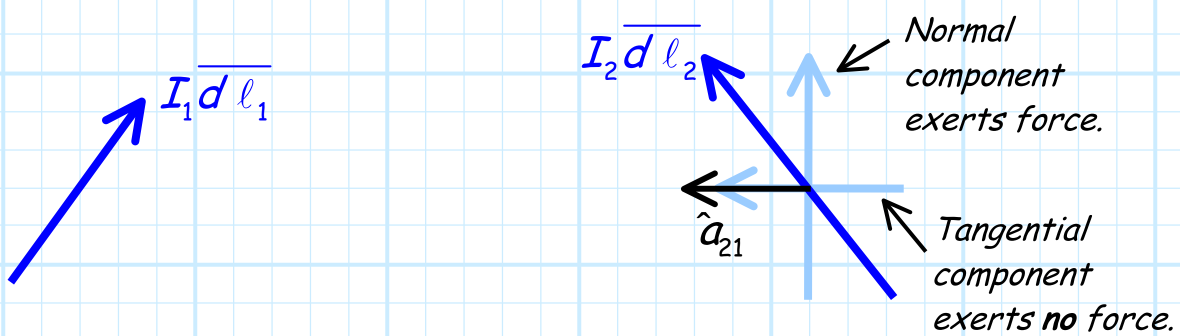
And thus:

$$d\mathbf{F}_1 = d\mathbf{F}_1^a + d\mathbf{F}_1^b$$

$$= -d\mathbf{F}_1^b + d\mathbf{F}_1^b = 0$$

In other words, if filament 2 **points** at filament 1, then the force on filament 1 is **zero**, **regardless** of the orientation of filament 1.

Another way of saying this is that **only** the component of $I_2\overline{d\ell_2}$ that is **orthogonal** to $\hat{a}_{21}$ can exert of force on filament 1.



Example 2: Filament 1 is **parallel** to filament 2

Therefore,

$$\overline{d\ell_1} \cdot \hat{a}_{21} = 0$$

so:

$$d\mathbf{F}_1^a = \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \hat{a}_{21}\right)\ \overline{d\ell_2} = 0$$

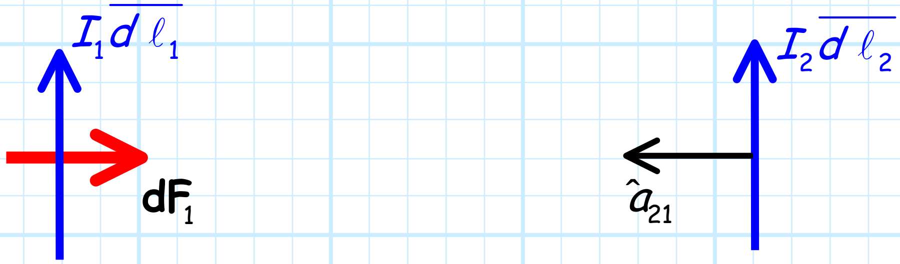
But,

$$\overline{d\ell_1} \cdot \overline{d\ell_2} \neq 0$$

therefore:

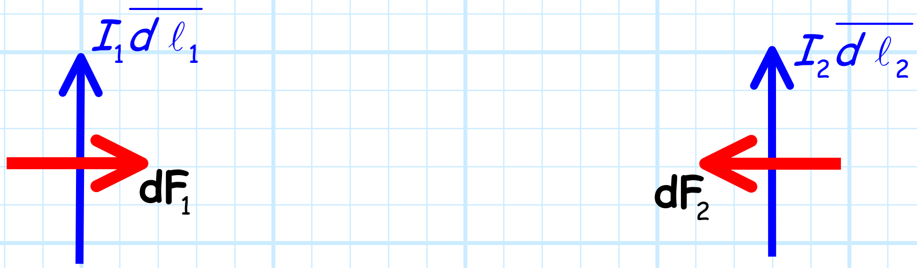
$$d\mathbf{F}_1^b = -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \overline{d\ell_2}\right)\ \hat{a}_{21} \neq 0$$

Thus, $d\mathbf{F}_1 = d\mathbf{F}_1^b$, applying a force in the direction $-\hat{a}_{21}$!



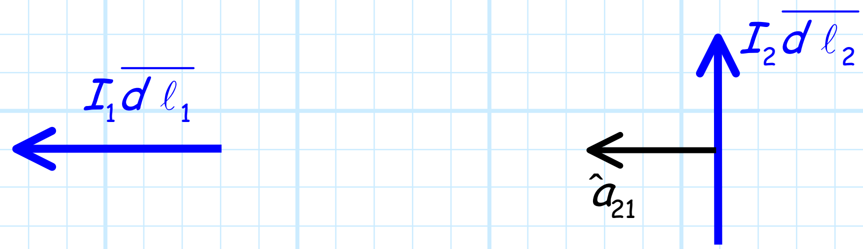
Filament 1 is **attracted** to filament 2 !

For the same reasons, filament 2 is attracted to filament 1:



But, we find that the two filaments **repel** if they point in **opposite** directions:

Example 3: Filament 1 is **parallel** to $\hat{a}_{21}$ and **orthogonal** to filament 2.



Therefore,

$$\overline{d\ell_1}\cdot\overline{d\ell_2}=0$$

so:

$$d\mathbf{F}_1^b=-\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1}\cdot\overline{d\ell_2}\right)\ \hat{a}_{21}=0$$

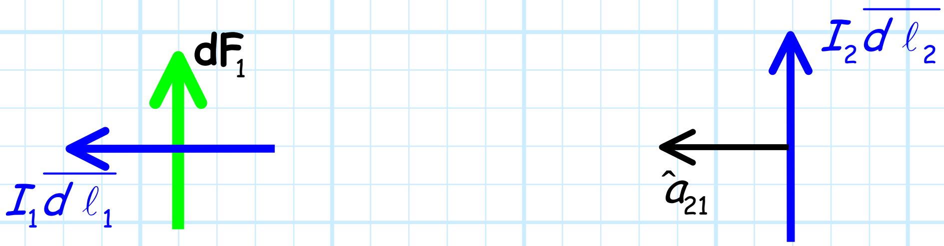
But,

$$\overline{d\ell_1}\cdot\hat{a}_{21}\neq 0$$

thus:

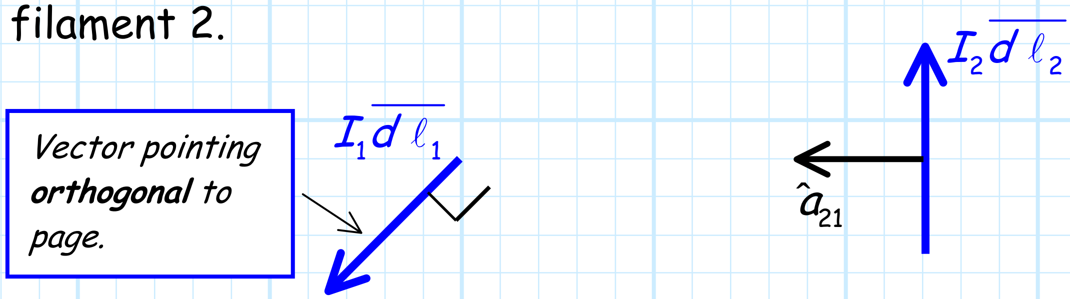
$$d\mathbf{F}_1^a=\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1}\cdot\hat{a}_{21}\right)\ \overline{d\ell_2}\neq 0$$

Therefore, $d\mathbf{F}_1 = d\mathbf{F}_1^a$, applying a force in the direction $\overline{d\ell_2}$:



Note however, the force on filament **2** is **zero** !

Example 4: Filament 1 is **orthogonal** to $\hat{a}_{21}$ and **orthogonal** to filament 2.



In this case, we find:

$$\overline{d\ell_1} \cdot \hat{a}_{21} = 0$$

so:

$$d\mathbf{F}_1^a = \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \hat{a}_{21}\right)\ \overline{d\ell_2} = 0$$

Likewise,

$$\overline{d\ell_1} \cdot \overline{d\ell_2} = 0$$

thus:

$$d\mathbf{F}_1^b = -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overline{d\ell_1} \cdot \overline{d\ell_2}\right)\ \hat{a}_{21} = 0$$

Therefore, the total force on filament 1 is **zero**:

$$d\mathbf{F}_1 = d\mathbf{F}_1^a + d\mathbf{F}_1^b = 0$$

For the same reasons, we find that the force on filament 2 due to filament 1 is **also zero** (i.e., $d\mathbf{F}_2 = 0$).