3-3 Current and Current Density (pp.63-68)

HO: Charge and Current

A. Volume Current Density

Q:

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HO: Surface Current Density

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Charge and Current

Say we have a conductor (e.g., wire) with $I = 1$ Ampere of current flowing through it.

Q: *What does this mean,* ***physically*** *?*

A: Current I simply describes the **rate** at which **net** charge passes through the wire cross-sectional surface S. For example, if a **net** charge ΔQ moves across surface S in some small amount of time Δt, we find that:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

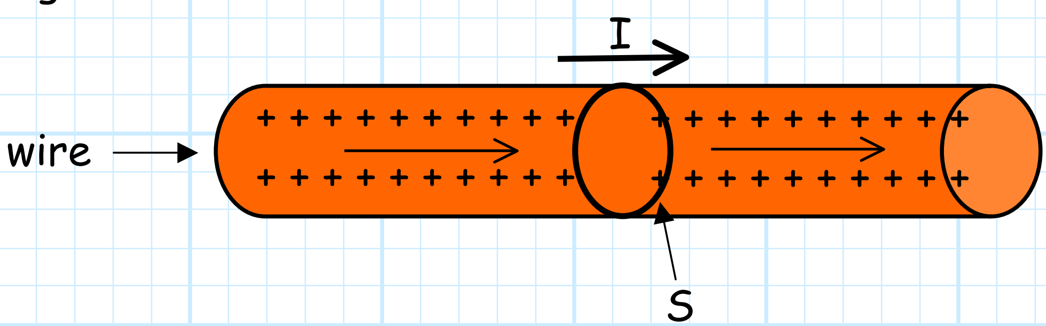
Thus, we find that 1 Amp means **+1.0 Coulomb** of net charge passes by a location on the wire each **second**, with the net charge in this case flowing from left to right.

Q: *The current is **positive**, does this mean that the current is made up of **positive** charge?*

A: **No!** Current generally consists of **both** positively and negatively charged particles.

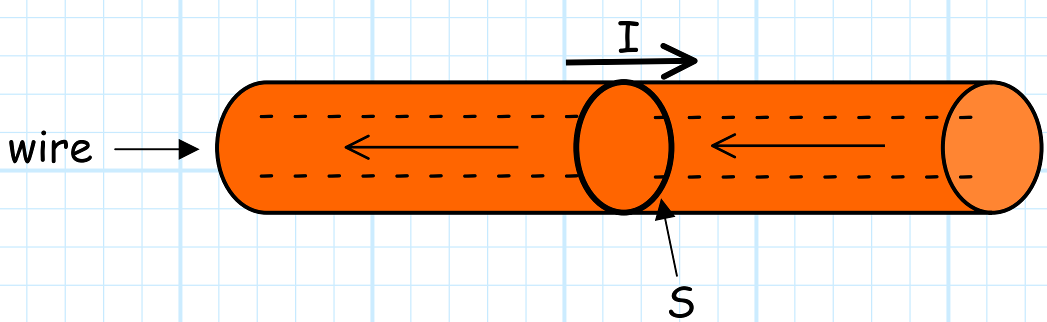
Remember, current is the **net** change in charge with respect to time.

For example, say **positive** charges are moving from **left to right** through the wire:



The current due to these charges is **positive**, as the total net charge on the right side of the surface is **increasing** with time.

That was pretty obvious, but here's the **tricky** part: say **negative** charges are moving from **right to left** through the wire (the **opposite** direction of that above).



Note in this case, the total charge on the right side of S is **again increasing** !

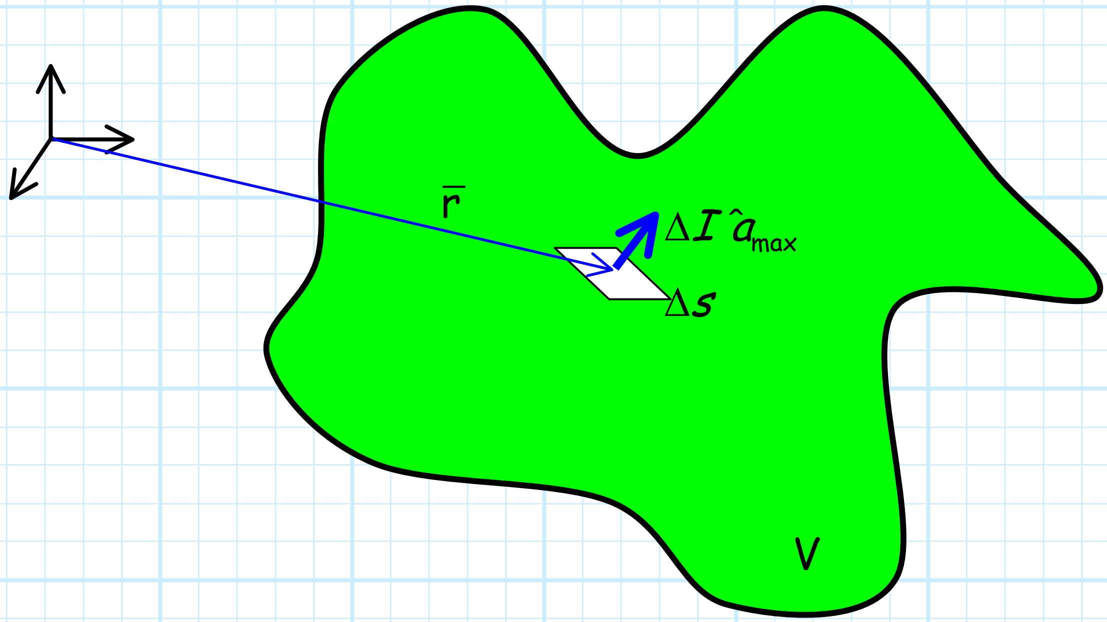
With the first case, the net charge was increasing because positive charges were entering the right side. For this case, the net charge on the right side is **also** increasing, but because negative charge is **leaving** the right side !
For reasons we shall learn about later, if positive charge moves one direction, then negative charge will generally move in the **opposite** direction. Therefore, total current is composed of charges moving in **both** directions:

$$I = I^{+} + I^{-}$$

Generally speaking, it **does not matter** (in fact we generally cannot tell) whether the particles that form a specific current are negative or positive—all that matters is the **net** change in charge across a surface.

Volume Current Density

Say at a given point $\bar{r}$ located in a volume V, charge is moving in **direction** $\hat{a}_{max}$.



Now, consider a **small surface** Δs that is centered at the point denoted by $\bar{r}$, and oriented such that it is orthogonal to unit vector $\hat{a}_{max}$. Since charge is moving across this small surface at some rate (coulombs/sec), we can define a **current** $\Delta I = \Delta Q / \Delta t$ that represents the current flowing through Δs.

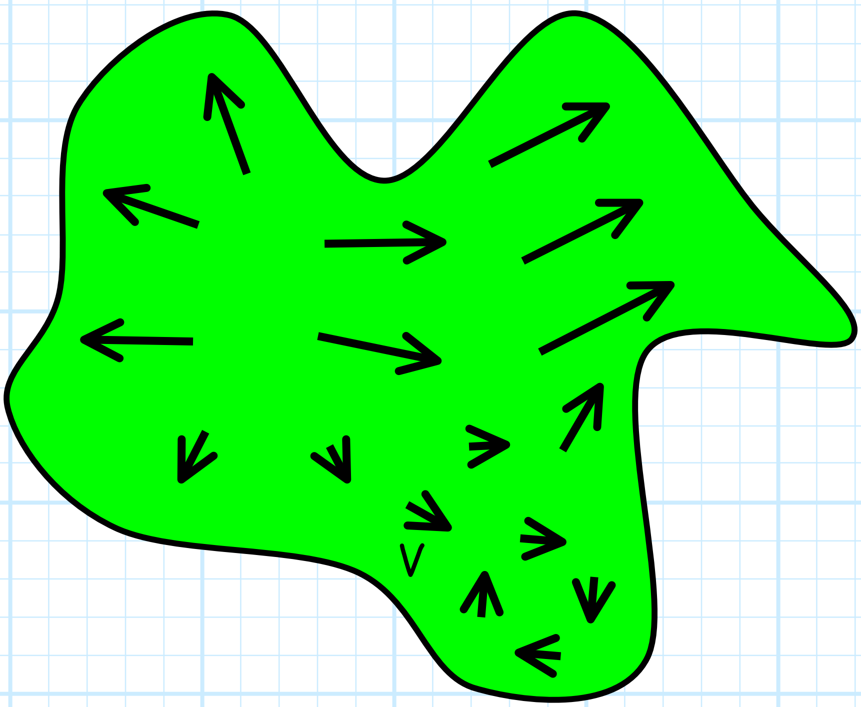
Note **vector** $\Delta I \hat{a}_{max}$ therefore represents both the **magnitude** (ΔI) and **direction** $\hat{a}_{max}$ of the current flowing through surface area Δs at point $\bar{r}$.

From this, we can define a **volume current density** $\mathbf{J}(\bar{r})$ at each and every point $\bar{r}$ in volume V by **normalizing** $\Delta I\ \hat{a}_{max}$ by dividing by the surface area Δs:

$$\mathbf{J}(\bar{r}) = \lim_{\Delta s \to 0} \frac{\Delta I\ \hat{a}_{max}}{\Delta s} \qquad \left[\frac{\text{Amps}}{\text{m}^2}\right]$$

The result is a **vector** field !

For example, current density $\mathbf{J}(\bar{r})$ might look like:



NOTE: The **unit** of **volume** current density is **current/area**; for example, A/m^2.

The Current I through Surface S

Given that we know volume current density $\mathbf{J}(\bar{r})$ throughout some volume, we can find the **total current** through **any** arbitrary **surface S** as:

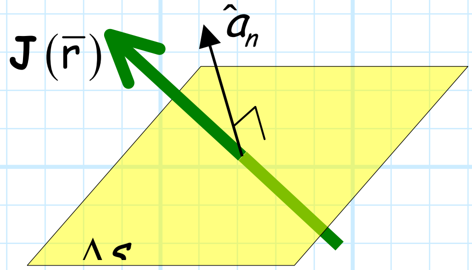
$$I = \iint_S \mathbf{J}(\bar{r}_s) \cdot \overline{ds} \qquad [Amps]$$

This integral is in the form of the **surface integral** we studied in Section 2-5.

Note the **integrand** has units of **current** (amps):

$$\mathbf{J}(\bar{r}_s) \cdot \overline{ds} = J_n(\bar{r}_s)\left|\overline{ds}\right| \qquad \left[\left(\frac{\text{Amps}}{\text{m}^2}\right)\left(\text{m}^2\right) = \text{Amps}\right]$$

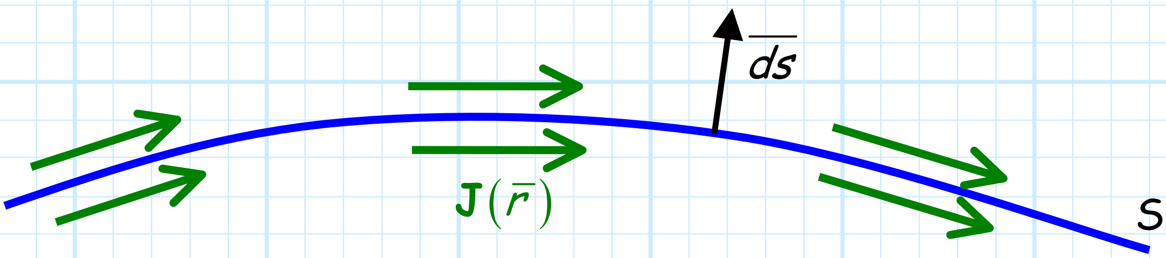
Physically, the value $\Delta I = \mathbf{J}(\bar{r}) \cdot \overline{ds}$ is the current flowing **through** the tiny differential surface Δs, located at point $\bar{r}$ on surface S.



Therefore if we **add** up (i.e., integrate) the current flowing through **each** and every differential surface element Δs that makes up surface S, we determine the **total** current I flowing **through** surface S.

Note the **sign** of current I is determined by the **direction** of differential surface vector $\overline{ds}$. For **example**, if I is **positive**, then the current is flowing **through** the surface in the direction of $\overline{ds}$.

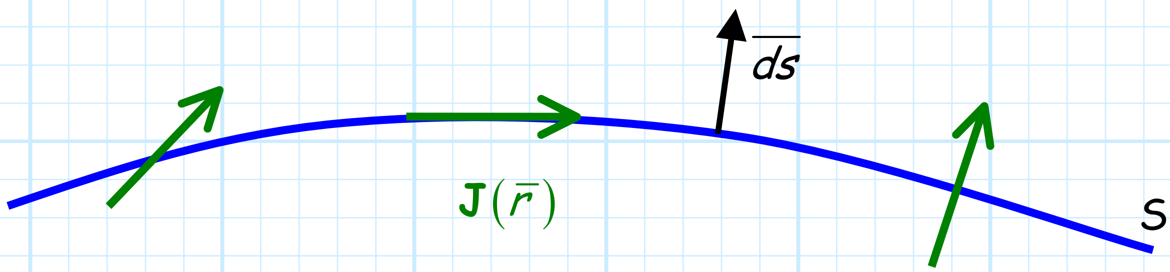
So, consider the case where $\mathbf{J}(\bar{r})$ describes current that is flowing **tangential** to **every point** on surface S. In other words, the current density has no **normal** component on the surface S !



As a result, we find that $\mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$ at **every** point on the surface, and therefore the surface **integral** results in I = 0.

This of course is **physically** the correct answer! Current is flowing **along** the surface, but none is flowing **through** it.

To get a **non-zero** amount of total current, the current density must have a **normal** component at **some** points on the surface.



For the case above, $I \neq 0$.

Q: *We know that if* $\mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$ *at all points on a surface, then the current flowing through the surface is zero (I=0).*

Is the ***converse*** *true? That is,* ***if*** *the the total current through a surface is* ***zero****, does that mean that the current density is* ***tangential*** *to the surface at* ***all*** *points?*

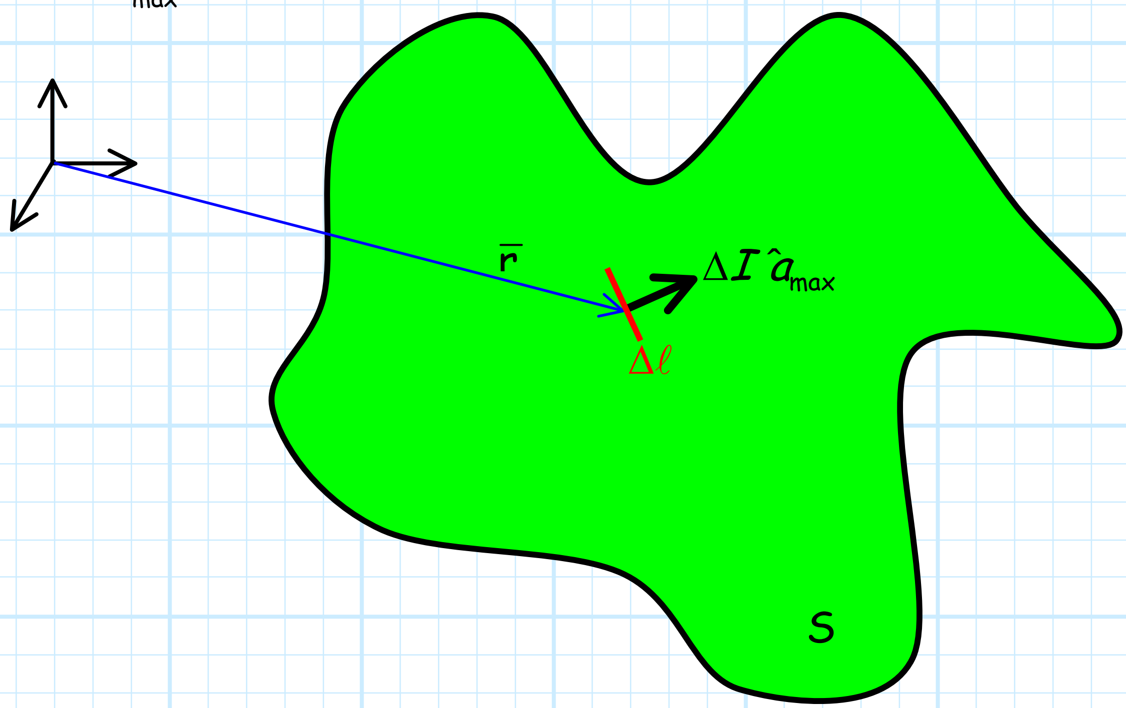
A:

Surface Current Density

Consider now the problem where we have moving **surface** charge $\rho_s(\bar{r})$.

The result is **surface** current!

Say at a given point $\bar{r}$ located on a surface S, charge is moving in **direction** $\hat{a}_{max}$.



Now, consider a **small length** of contour $\Delta\ell$ that is centered at point $\bar{r}$, and oriented such that it is orthogonal to unit vector $\hat{a}_{max}$. Since charge is moving across this small length, we can define a **current** ΔI that represents the current flowing across $\Delta\ell$.

Note **vector** $\Delta I\, \hat{a}_{max}$ therefore represents both the **magnitude** (ΔI) and **direction** $\hat{a}_{max}$ of the current flowing across contour $\Delta\ell$ at point $\bar{r}$.

From this, we can define a **surface current density** $\mathbf{J}_s(\bar{r})$ at every point $\bar{r}$ on surface S by **normalizing** $\Delta I\, \hat{a}_{max}$ by dividing by the length $\Delta\ell$:

$$\mathbf{J}_s(\bar{r}) = \lim_{\Delta\ell \to 0} \frac{\Delta I\, \hat{a}_{max}}{\Delta\ell} \qquad \left[\frac{\text{Amps}}{\text{m}}\right]$$

The result is a **vector** field !

NOTE: *The unit of* ***surface*** *current density is current/****length****; for example, A/m.*

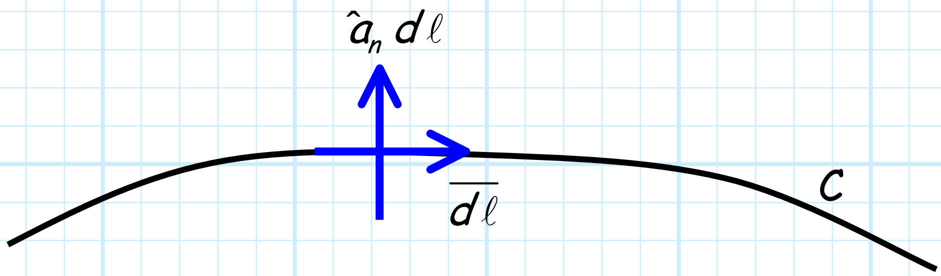
Given that we know surface current density $\mathbf{J}_s(\bar{r})$ throughout some volume, we can find the total **current** across **any** arbitrary **contour *C*** as:

$$I = \int_C \mathbf{J}_s(\bar{r}) \cdot \hat{a}_n\, d\ell$$

This looks very much like the contour integral we studied in the previous chapter. However, there is one **big** difference!

The differential vector $\hat{a}_n \, d\ell$ is a vector that tangential to **surface** S (i.e., it lies on surface S), but is **normal** to contour C!

This of course is the **opposite** of the differential vector $\overline{d\ell}$ in that $\overline{d\ell}$ lies **tangential** to the contour:



As a result, we find that $\overline{d\ell} \cdot \hat{a}_n \, d\ell = 0$. However, note the **magnitude** of each vector is identical:

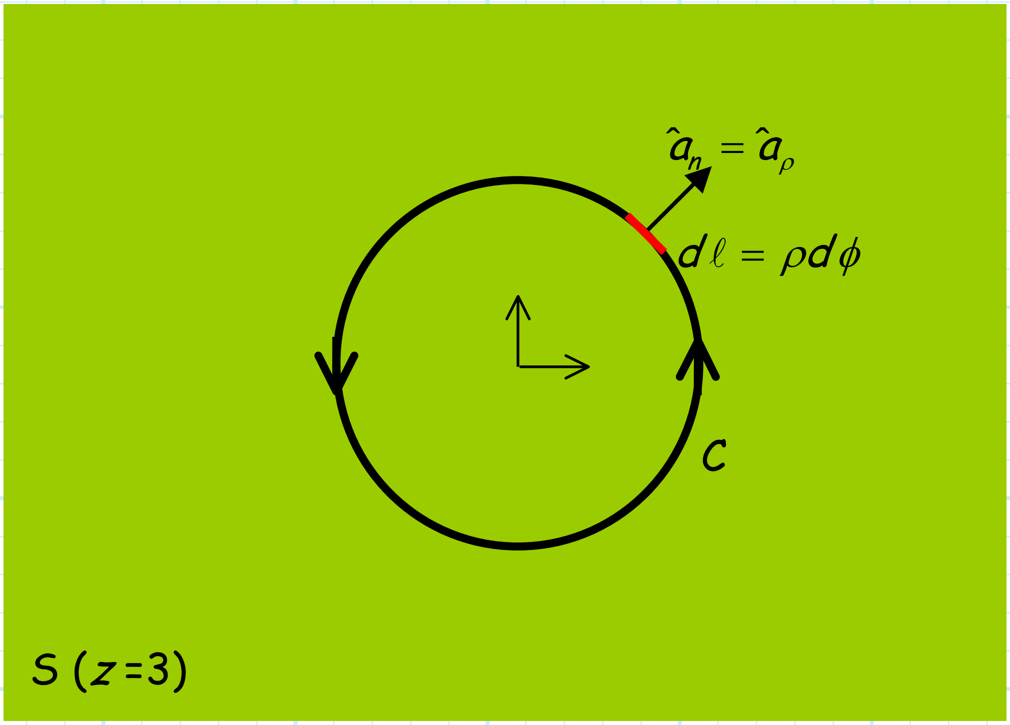
$$\left|\overline{d\ell}\right| = \left|\hat{a}_n \, d\ell\right| = d\ell$$

For example, consider the planar surface z=3. On this surface is a contour that is a **circle**, radius 2, centered around the z-axis.

For the contour integrals we studied in Section 2-5, we would use:

$$\overline{d\ell} = \hat{a}_\phi \, \rho \, d\phi$$

However, to determine the total current flowing across the contour, we use $\hat{a}_n = \hat{a}_\rho$ and $d\ell = \rho \, d\phi$. Note the **directions** of these two differential vectors are **different**, but their **magnitudes** are the **same**.



The integral for determining the **total** current flowing from **inside** the circle to **outside** the circle is therefore:

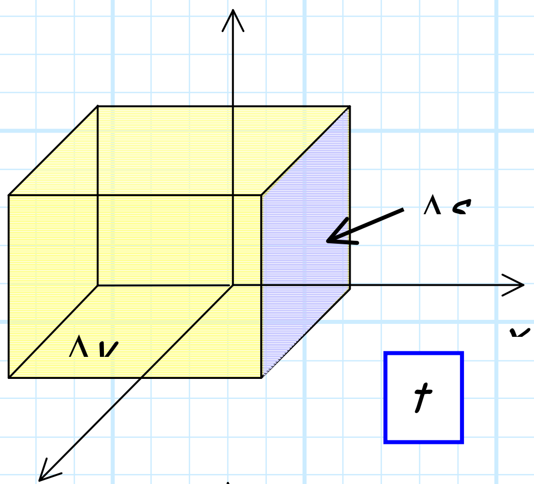
$$
\begin{aligned}
I &= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_n \, d\ell \\
&= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho \, \rho \, d\phi \\
&= \int_0^{2\pi} \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho 2 \, d\phi
\end{aligned}
$$

Charge Velocity and Current Density

Consider a **small** volume (Δv) filled with charge Q.

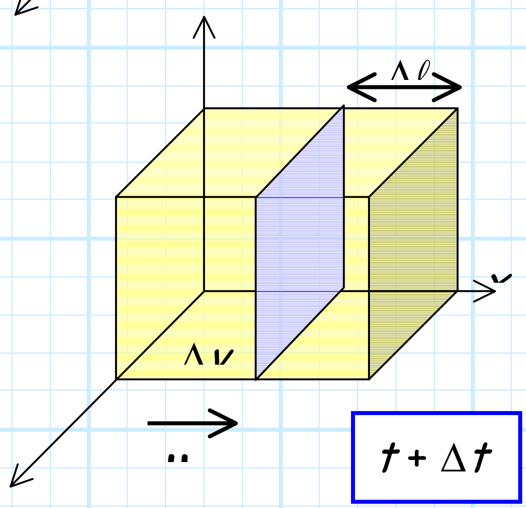
If the charge is **uniformly** distributed, then the **charge density** is:

$$\rho_v(\bar{\mathbf{r}}) = \frac{Q}{\Delta v}$$



Say these charges are **moving** at velocity $\mathbf{u} = u_x\,\hat{a}_x$. Then, in a small **time** Δt, the charged particles will have moved in the x-direction a **distance** $\Delta \ell$:

$$\Delta \ell = u_x\,\Delta t$$



Q: *How much **charge** ΔQ moves across **surface** Δs in **time** Δt ?*

A: The amount is **equal** to the charge occupying **volume** $\Delta s\,\Delta \ell$:

$$\Delta Q = \rho_v(\bar{r})(\Delta s \, \Delta \ell)$$

But remember, $\Delta \ell = u_x \, \Delta t$. Therefore:

$$\Delta Q = \rho_v(\bar{r}) \, u_x \, \Delta s \, \Delta t$$

And dividing by Δt:

$$\frac{\Delta Q}{\Delta t} = \rho_v(\bar{r}) \, u_x \, \Delta s$$

Hey! Charge divided by time is equal to **current** !

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v(\bar{r}) \, u_x \, \Delta s$$

The current ΔI is the current flowing **through** the small surface Δs. We can therefore determine the **current density** on this surface:

$$J_x = \frac{\Delta I}{\Delta s} = \rho_v(\bar{r}) \; u_x$$

In other words, current density is equal to the **product** of the charge density and the charge velocity. In general, we can say:

$$\mathbf{J}(\bar{r}) = \rho_v(\bar{r}) \, \mathbf{u}(\bar{r})$$

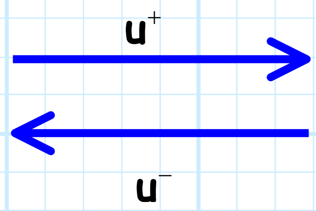
where $\mathbf{u}(\bar{r})$ is a vector field that describes the **velocity** of the moving charge at every point $\bar{r}$.

IMPORTANT NOTE! The velocity of charge is **NOT** the speed of light! In fact, charge velocity is generally **nowhere** near c = $3x10^8$ m/sec (its more like $3x10^{-2}$ m/sec!).

Charge velocity is generally dependent on the **type** of particles that carry the charge (e.g., free electrons, positive ions).

For example, we can denote $\mathbf{u}^+$ the velocity of **positively** charged particles, while $\mathbf{u}^-$ denotes the velocity of **negatively** charged particles.

We find that typically, $\mathbf{u}^+$ and $\mathbf{u}^-$ point in **opposite** directions!



and the velocities will have **unequal** magnitudes:

$$|\mathbf{u}^+| \neq |\mathbf{u}^-|$$

The total current density can therefore be expressed as:

$$\begin{aligned}\mathbf{J}(\bar{r}) &= \mathbf{J}^+(\bar{r}) + \mathbf{J}^-(\bar{r}) \\ &= \rho_v^+(\bar{r})\,\mathbf{u}^+(\bar{r}) + \rho_v^-(\bar{r})\,\mathbf{u}^-(\bar{r})\end{aligned}$$

Q: *So, $\mathbf{J}^{+}(\bar{r})$ and $\mathbf{J}^{-}(\bar{r})$ must point in opposite directions, since $\mathbf{u}^{+}(\bar{r})$ and $\mathbf{u}^{-}(\bar{r})$ point in opposite directions ?*

A: NO! It is true that the charges flow in opposite **directions**, but the charges also have opposite **signs** ! Recall $\rho_v^{+}(\bar{r}) > 0$ and $\rho_v^{-}(\bar{r}) < 0$, therefore, vectors $\mathbf{J}^{+}(\bar{r}) = \rho_v^{+}(\bar{r})\mathbf{u}^{+}(\bar{r})$ and $\mathbf{J}^{-}(\bar{r}) = \rho_v^{-}(\bar{r})\mathbf{u}^{-}(\bar{r})$ each typically point in the **same** direction!

Remember that, for example, if positive charge is moving **left** and negative charge is moving **right**, then **both** result in **current** flowing toward the **left**.