4-2 Maxwell's Equations for Electrostatics

Reading Assignment: *pp. 88-90*

Q:

A:

HO: The Electrostatic Equations

HO: The Integral Form of Electrostatics

The Electrostatic Equations

If we consider the **static** case (i.e., constant with time) of Maxwell's Equations, we find that the **time derivatives** of the electric field and magnetic flux density are **zero**:

$$\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t} = 0$$

Thus, Maxwell's equations for **static fields** become:

$$\nabla \times \mathbf{E}(\bar{r}) = 0$$

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \cdot \mathbf{B}(\bar{r}) = 0$$

Look at what has happened! For the static case (but **just** for the static case!), Maxwell's equations "**decouple**" into two **independent** sets of two equations.

The first set involves electric field $\mathbf{E}(\bar{r})$ and charge density $\rho_v(\bar{r})$ only. These are called the **electrostatic equations** in **free-space**:

$$\nabla x \mathbf{E}(\bar{r}) = 0$$

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

These are the **electrostatic equations** for free space (i.e., a vacuum).

Note that the **static** electric field is a **conservative** vector field (do you see why ?)!

This of course means that **everything** we know about a conservative field is true **also** for the **static** field $\mathbf{E}(\bar{r})$.

Essentially, **this** is what the electrostatic equations **tell** us:

1) The **static** electric field is **conservative**.

2) The **source** of the static field is **charge**:

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

In other words, the static electric field $\mathbf{E}(\bar{r})$ **diverges** from (or **converges** to) charge!

Chapters 4, 5, and 6 deal only with **electrostatics** (i.e., static electric fields produced by static charge densities).

In chapters 7, 8, and 9, we will study **magnetostatics**, which considers the **other** set of static differential equations:

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \cdot \mathbf{B}(\bar{r}) = 0$$

These equations are called the **magnetostatic equations** in free-space, and relate the static **magnetic flux density** $\mathbf{B}(\bar{r})$ to the static **current density** $\mathbf{J}(\bar{r})$.

The Integral Form of Electrostatics

We know from the **static** form of Maxwell's equations that the vector field $\nabla x \mathbf{E}(\bar{r})$ is zero at every point $\bar{r}$ in space (i.e., $\nabla x \mathbf{E}(\bar{r})=0$). Therefore, **any** surface integral involving the vector field $\nabla x \mathbf{E}(\bar{r})$ will likewise be zero:

$$\iint_S \nabla x \mathbf{E}(\bar{r}) \cdot \overline{ds} = 0$$

But, using **Stokes' Theorem**, we can also write:

$$\iint_S \nabla x \mathbf{E}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

Therefore, the equation:

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

is the **integral form** of the equation:

$$\nabla x \mathbf{E}(\bar{r}) = 0$$

Of course, both equations just indicate that the **static electric field $\mathbf{E}(\bar{r})$** is a **conservative** field!

Likewise, we can take a volume integral over both sides of the electrostatic equation $\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})/\varepsilon_0$:

$$\iiint_V \nabla \cdot \mathbf{E}(\bar{\mathbf{r}})\, dv = \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{\mathbf{r}})\, dv$$

But wait! The left side can be rewritten using the **Divergence Theorem**:

$$\iiint_V \nabla \cdot \mathbf{E}(\bar{\mathbf{r}})\, dv = \oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds}$$

And, we know that the volume integral of the charge density is equal to the **charge enclosed** in volume V:

$$\iiint_V \rho_v(\bar{\mathbf{r}})\, dv = Q_{enc}$$

Therefore, we can write an equation known as **Gauss's Law**:

$$\oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

Gauss's Law

This is the **integral form** of the equation $\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})/\varepsilon_0$.

What **Gauss's Law** says is that we can determine the total amount of **charge enclosed** within some **volume** V by simply integrating the electric field on the surface S surrounding volume V.

Summarizing, the **integral form** of the electrostatic equations are:

$$\oint_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} = 0 \qquad \oiint_S \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{ds} = \frac{Q}{\varepsilon_0}$$

Note that these equations do **not** amend or extend what we already know about the static electric field, but are simply an **alternative** way of expressing the **point** form of the electrostatic equations:

$$\nabla \times \mathbf{E}(\bar{\mathbf{r}}) = 0$$

$$\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

We **sometimes** use the **point** form of the electrostatic equations, and we sometimes use the **integral** form—it all depends on which form is more applicable to the problem we are attempting to solve!