4-3 E-field Calculations using Coulomb's Law

Reading Assignment: *pp. 93-98*

The Uniform, Infinite Line Charge

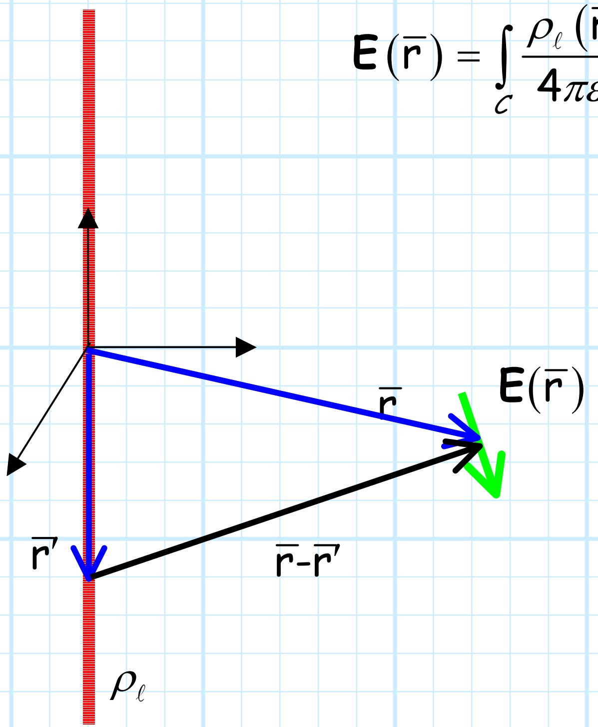
Consider an **infinite** line of charge lying along the *z*-axis. The charge density along this line is a **constant** value of ρ_ℓ C/m.

Q: *What electric field* $\mathbf{E}(\bar{\mathbf{r}})$ *is produced by* ***this*** *charge distribution?*

A: Apply **Coulomb's Law**!

We know that for a **line charge** distribution that:

$$\mathbf{E}(\bar{\mathbf{r}}) = \int_C \frac{\rho_\ell(\bar{\mathbf{r}}')}{4\pi\varepsilon_0} \frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}'}{\left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|^3} d\ell'$$



Q: *Yikes! How do we evaluate **this** integral?*

A: Don't panic! **You** know how to evaluate this integral. Let's break up the process into **smaller steps**.

Step 1: Determine $d\ell'$

The differential element $d\ell'$ is just the **magnitude** of the differential line element we studied in chapter 2 (i.e., $d\ell' = \left|\overline{d\ell'}\right|$). As a result, we can easily integrate over **any** of the seven contours we discussed in chapter 2.

The contour in this problem **is** one of those! It is a line parallel to the *z*-axis, defined as $x'=0$ and $y'=0$. As a result, we use for $d\ell'$:

$$d\ell' = \left|\hat{a}_z\, dz'\right| = dz'$$

Step 2: Determine the **limits of integration**

This is easy! The line charge is **infinite**. Therefore, we integrate from $z' = -\infty$ to $z' = \infty$.

Step 3: Determine the **vector** $\bar{r}-\bar{r}'$.

Since for all charge $x'=0$ and $y'=0$, we find:

$$\begin{aligned}\bar{r}-\bar{r}' &= \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) - \left(x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z\right)\\ &= \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) - z'\hat{a}_z\\ &= x\hat{a}_x + y\hat{a}_y + \left(z - z'\right)\hat{a}_z\end{aligned}$$

Step 4: Determine the **scalar** $|\bar{r}-\bar{r}'|^3$

Since $|\bar{r}-\bar{r}'| = \sqrt{x^2 + y^2 + (z-z')^2}$, we find:

$$|\bar{r}-\bar{r}'|^3 = \left[x^2 + y^2 + (z-z')^2\right]^{3/2}$$

Step 5: Time to integrate !

$$\begin{aligned}
\mathbf{E}(\bar{r}) &= \int_C \frac{\rho_\ell(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}\, d\ell' \\
&= \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \rho_\ell \frac{x\,\hat{a}_x + y\,\hat{a}_y + (z-z')\,\hat{a}_z}{\left[x^2 + y^2 + (z-z')^2\right]^{3/2}}\, dz' \\
&= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{x\,\hat{a}_x + y\,\hat{a}_y + (z-z')\,\hat{a}_z}{\left[x^2 + y^2 + (z-z')^2\right]^{3/2}}\, dz' \\
&= \frac{\rho_\ell\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{\left[x^2 + y^2 + (z-z')^2\right]^{3/2}} \\
&\quad + \frac{\rho_\ell\,\hat{a}_z}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{(z-z')\, dz'}{\left[x^2 + y^2 + (z-z')^2\right]^{3/2}} \\
&= \frac{\rho_\ell\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{4\pi\varepsilon_0} \frac{2}{x^2 + y^2} + 0 \\
&= \frac{\rho_\ell}{2\pi\varepsilon_0} \frac{\left(x\,\hat{a}_x + y\,\hat{a}_y\right)}{x^2 + y^2}
\end{aligned}$$

This result, however, is best expressed in **cylindrical coordinates**:

$$\frac{x\hat{a}_x + y\hat{a}_y}{x^2 + y^2} = \frac{\rho\cos\phi\,\hat{a}_x + \rho\sin\phi\,\hat{a}_y}{\rho^2}$$

$$= \frac{\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y}{\rho}$$

And with cylindrical **base vectors**:

$$\begin{aligned}\frac{\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y}{\rho} &= \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_\rho + \sin\phi\,\hat{a}_y\cdot\hat{a}_\rho\right)\hat{a}_\rho \\ &+ \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_\phi + \sin\phi\,\hat{a}_y\cdot\hat{a}_\phi\right)\hat{a}_\phi \\ &+ \frac{1}{\rho}\left(\cos\phi\,\hat{a}_x\cdot\hat{a}_z + \sin\phi\,\hat{a}_y\cdot\hat{a}_z\right)\hat{a}_z \\ &= \frac{1}{\rho}\left(\cos^2\phi + \sin^2\phi\right)\hat{a}_\rho \\ &+ \frac{1}{\rho}\left(-\cos\phi\sin\phi + \sin\phi\cos\phi\right)\hat{a}_\phi \\ &+ \frac{1}{\rho}\left(\cos\phi\,(0) + \sin\phi\,(0)\right)\hat{a}_z \\ &= \frac{\hat{a}_\rho}{\rho}\end{aligned}$$

As a result, we can write the **electric field** produced by an **infinite line charge** with constant density ρ_ℓ as:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_\ell}{2\pi\varepsilon_0}\frac{\hat{a}_\rho}{\rho}$$

Note what this means. Recall unit vector $\hat{a}_\rho$ is the direction that **points away from** the z-axis. In other words, the electric field produced by the uniform line charge points away from the line charge, just like the electric field produced by a point charge likewise points away from the charge.

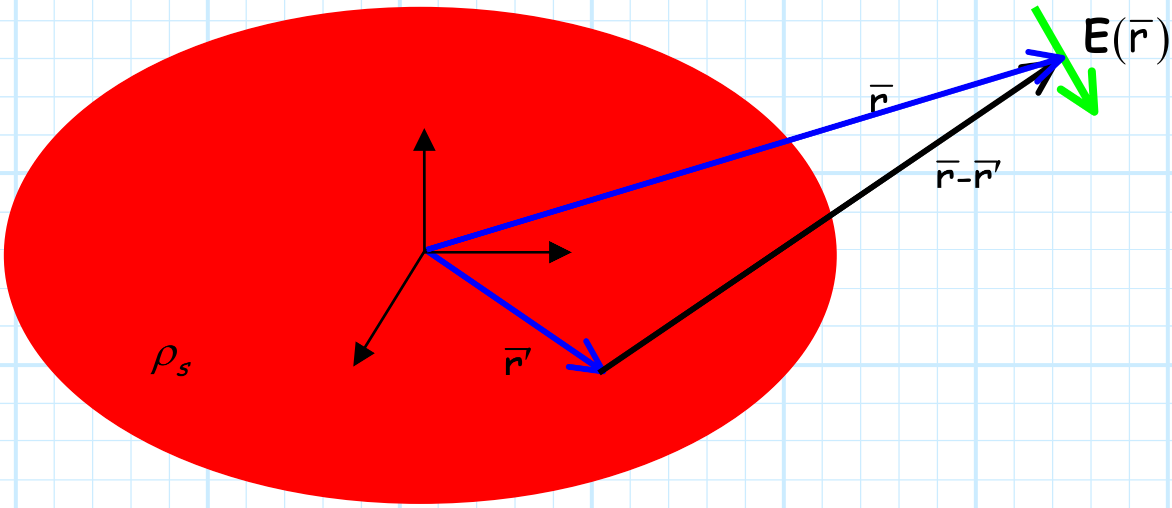
It is apparent that the electric field in the static case appears to **diverge** from the location of the charge. And, this is exactly what Maxwell's equations (**Gauss's Law**) says will happen ! i.e.,:

$$\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

Note the **magnitude** of the electric field is proportional to $1/\rho$, therefore the electric field **diminishes** as we get further from the line charge. Note however, the electric field does not diminish as **quickly** as that generated by a point charge. Recall in that case, the magnitude of the electric field diminishes as $1/r^2$.

The Uniform Disk of Charge

Consider a **disk** radius *a*, centered at the origin, and lying entirely on the $z=0$ plane.



This disk contains **surface charge**, with density of ρ_s C/m^2. This density is **uniform** across the disk.

Let's find the **electric field** generated by this charge disk!

From **Coulomb's Law**, we know:

$$\mathbf{E}(\bar{\mathbf{r}}) = \iint_S \frac{\rho_s(\bar{\mathbf{r}}')}{4\pi\varepsilon_0} \frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}'}{\left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|^3}\, ds'$$

Step 1: Determine ds'

This disk can be described by the equation $z' = 0$. That is, every point on the disk has a cordinate value z' that is equal to zero.

This is **one** of the surfaces we examined in chapter 2. The **differential surface element** for that surface, you recall, is:

$$ds' = ds_z = \rho' \, d\rho' \, d\phi'$$

Step 2: Determine the **limits of integration** .

Note over the surface of the disk, ρ' changes from 0 to radius a, and ϕ' changes from 0 to 2π. Therefore:

$$0 < \rho' < a \qquad 0 < \phi' < 2\pi$$

Step 3: Determine **vector** $\bar{r}-\bar{r}'$.

We know that $z' = 0$ for all charge, therefore we can write:

$$\begin{aligned}\bar{r}-\bar{r}' &= \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) - \left(x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z\right) \\ &= \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) - \left(x'\hat{a}_x + y'\hat{a}_y\right) \\ &= (x-x')\hat{a}_x + (y-y')\hat{a}_y + z\hat{a}_z\end{aligned}$$

Since the primed coordinates in ds' are expressed in **cylindrical** coordinates, we convert the coordinates to get:

$$
\begin{aligned}
\bar{r}-\bar{r}' &= \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) - \left(x'\hat{a}_x + y'\hat{a}_y\right) \\
&= \left(x - x'\right)\hat{a}_x + \left(y - y'\right)\hat{a}_y + z\,\hat{a}_z \\
&= \left(x - \rho'\cos\phi'\right)\hat{a}_x + \left(y - \rho'\sin\phi'\right)\hat{a}_y + z\,\hat{a}_z
\end{aligned}
$$

Step 4: Determine $\left|\bar{r}-\bar{r}'\right|^3$

We find that:

$$
\left|\bar{r}-\bar{r}'\right|^3 = \left[\left(x-\rho'\cos\phi'\right)^2 + \left(y - \rho'\sin\phi'\right)^2 + z^2\right]^{3/2}
$$

Step 5: Time to integrate !

$$
\begin{aligned}
\mathbf{E}(\bar{r}) &= \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0}\frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3}\,ds' \\
&= \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{a}\frac{\left(x-\rho'\cos\phi'\right)\hat{a}_x + \left(y-\rho'\sin\phi'\right)\hat{a}_y + z\,\hat{a}_z}{\left[\left(x-\rho'\cos\phi'\right)^2 + \left(y-\rho'\sin\phi'\right)^2 + z^2\right]^{3/2}}\,\rho' d\rho' d\phi'
\end{aligned}
$$

Yikes! What a **mess**! To **simplify** our integration let's determine the electric field $\mathbf{E}(\bar{r})$ along the ***z*-axis** only. In other words, set x = 0 and y = 0.

$$\mathbf{E}(x=0,y=0,z) = \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3} \, ds'$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^a \frac{(0-\rho'\cos\phi')\hat{a}_x + (0-\rho'\sin\phi')\hat{a}_y - z\hat{a}_z}{\left[(0-\rho'\cos\phi')^2 + (0-\rho'\sin\phi')^2 + z^2\right]^{3/2}} \rho' d\rho' d\phi'$$

$$= \frac{-\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^a \frac{(\rho'\cos\phi')\hat{a}_x + (\rho'\sin\phi')\hat{a}_y - z\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}} \rho' d\rho' d\phi'$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \hat{a}_x \int_0^{2\pi}\int_0^a \frac{(\rho'\cos\phi')\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$+ \frac{-\rho_s}{4\pi\varepsilon_0} \hat{a}_y \int_0^{2\pi}\int_0^a \frac{(\rho'\sin\phi')\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

$$+ \frac{-\rho_s}{4\pi\varepsilon_0} \hat{a}_z \int_0^{2\pi}\int_0^a \frac{z\, \rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}}$$

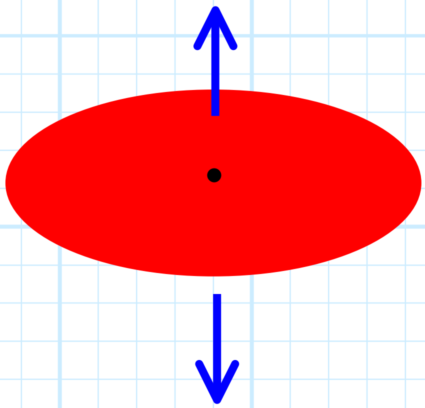
Note that since:

$$\int_0^{2\pi} \sin\phi \, d\phi = 0 = \int_0^{2\pi} \cos\phi \, d\phi$$

The first two terms (E_x and E_y) are equal to zero. Integrating the last term, we get:

$$\mathbf{E}(x=0,y=0,z) = \begin{cases} \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z\left[1 - \dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z > 0 \\[2ex] \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z\left[-1 - \dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z < 0 \end{cases}$$

From this expression, we can conclude **two** things. The first is that **above** the disk (z > 0), the electric field points in the direction $\hat{a}_z$, and below the disk (z < 0), it points in the direction $-\hat{a}_z$.



What a surprise (not)! The electric field **points away** from the charge. It appears to be **diverging** from the charged disk (as predicted by Gauss's Law).

Likewise, it is evident that as we move further and **further from** the disk, the electric field will **diminish**. In fact, as distance z goes to **infinity**, the magnitude of the electric field approaches **zero**. This of course is similar to the **point** or **line** charge; as we move an infinite distance away, the electric field diminishes to **nothing**.

An Infinite Charge Plane

Say that we have a **very large** charge disk. So large, in fact, that its radius a approaches **infinity** !

Q: *What electric field is created by this infinite plane?*

A: We **already** know! Just evaluate the charge disk solution for the case where the disk **radius** a is **infinity**.

In other words:

$$\lim_{a\to\infty} \mathbf{E}(x=0,y=0,z) = \begin{cases} \hat{a}_z \dfrac{\rho_s}{2\varepsilon_0}\left[1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z>0 \\ \\ \hat{a}_z \dfrac{\rho_s}{2\varepsilon_0}\left[-1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{if } z<0 \end{cases}$$

$$= \begin{cases} \dfrac{\rho_s}{2\varepsilon_0}\,\hat{a}_z & \text{if } z>0 \\ \\ \dfrac{-\rho_s}{2\varepsilon_0}\,\hat{a}_z & \text{if } z<0 \end{cases}$$

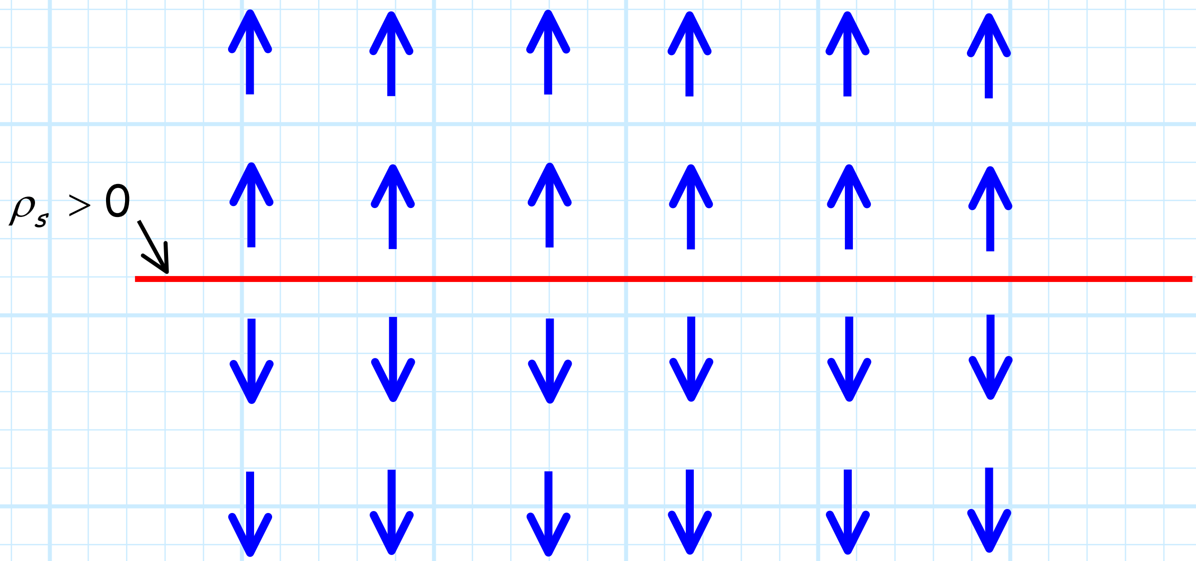
Therefore, the electric field produced by an **infinite charge plane**, with surface charge density ρ_s, is:

$$\mathbf{E}(\bar{r}) = \begin{cases} \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z & \text{if } z > 0 \\[2ex] \dfrac{-\rho_s}{2\varepsilon_0}\hat{a}_z & \text{if } z < 0 \end{cases}$$

Think about what **this** says!

* First, we note that the electric field **points away** from the plane if ρ_s is positive, and toward the plane if ρ_s is negative.

* Second, we notice that the magnitude of the electric field is a **constant**—the magnitude is **independent** of the distance from the infinite plane!



The reason for this result is, that no matter how far you are (i.e., $|z|$) from the infinite charge plane, you remain **infinitely close** to plane, when **compared** to its radius a.

We will find these results are useful when we study the behavior of a parallel plate **capacitor**.