6-4 Energy Storage in Capacitors

Reading Assignment: pp. 200-201

Q:

A: HO: Energy Storage in Capacitors
Energy Storage in Capacitors

Recall in a parallel plate capacitor, a surface charge distribution $\rho_{s+}(\vec{r})$ is created on one conductor, while charge distribution $\rho_{s-}(\vec{r})$ is created on the other.

**Q:** How much energy is stored by these charges?

**A:** Use the results of Section 6-3.

We learned that the energy stored by a charge distribution is:

$$W_e = \frac{1}{2} \iiint_V \rho_v(\vec{r}) V(\vec{r}) \, dV$$

The equivalent equation for surface charge distributions is:
\[ W_e = \frac{1}{2} \iiint_{\bar{S}} \rho_s (\bar{r}) V (\bar{r}) \, ds \]

For the parallel plate capacitor, we must integrate over both plates:

\[ W_e = \frac{1}{2} \iiint_{\bar{S}_+} \rho_{s+} (\bar{r}) V (\bar{r}) \, ds + \frac{1}{2} \iiint_{\bar{S}_-} \rho_{s-} (\bar{r}) V (\bar{r}) \, ds \]

But on the top plate (i.e., \( S_+ \)), we know that:

\[ V (z=-d) = V_0 \]

while on the bottom (i.e., \( S_- \)):

\[ V (z=0) = 0 \]

Therefore:

\[ W_e = \frac{V_0}{2} \iiint_{\bar{S}_+} \rho_{s+} (\bar{r}) \, ds + \frac{0}{2} \iiint_{\bar{S}_-} \rho_{s-} (\bar{r}) \, ds \]

\[ = \frac{V_0}{2} \iiint_{\bar{S}_+} \rho_{s+} (\bar{r}) \, ds \]

But, the remaining surface integral we know to be charge \( Q \):

\[ Q = \iiint_{\bar{S}_+} \rho_{s+} (\bar{r}) \, ds \]

Therefore, we find:

\[ W_e = \frac{1}{2} V_0 Q \]

But recall that:

\[ Q = CV \]
where $V$ is the potential difference between the two conductors (i.e., $V = V_0$).

Combining these two equations, we find:

$$W_e = \frac{1}{2} V_0 \frac{Q}{C}$$

$$= \frac{1}{2} V_0 (C V)$$

$$= \frac{1}{2} C V^2$$

The above equation shows that the energy stored within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

Recall that we also can determine the stored energy from the fields within the dielectric:

$$W_e = \frac{1}{2} \int \int \int_D \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) \, dV$$

Since the fields within the capacitor are approximately:

$$\mathbf{E}(\mathbf{r}) = \frac{V}{d} \hat{a}_z$$

$$\mathbf{D}(\mathbf{r}) = \frac{\varepsilon V}{d} \hat{a}_z$$

we find:
where the volume of the dielectric is simply the plate surface area $S$ times the dielectric thickness $d$:

$$\text{Volume} = S \ d$$

Resulting in the expression:

$$W_e = \frac{1}{2} \left( \frac{\varepsilon V^2}{d^2} \right) (S \ d) = \frac{1}{2} \frac{\varepsilon S}{d} V^2$$

Recall, however, that the capacitance of a parallel plate capacitor is:

$$C = \frac{\varepsilon S}{d}$$

Therefore:

$$W_e = \frac{1}{2} \frac{\varepsilon S}{d} V^2 = \frac{1}{2} CV^2$$

The same result as before!