9-2 Faraday's Law of Induction

Reading Assignment: *pp. 277-286*

Now let's consider **time-varying** fields!

Specifically, we consider what occurs when a **magnetic flux density** is **not** a constant with time (i.e., $\mathbf{B}(\bar{r},t)$).

→ Maxwell's Equations **"recouple"**, so that the **electric field** and magnetic flux density are **related**.

Specifically, a time varying magnetic field is the source of a new, **solenoidal electric field**!

HO: Faraday's Law

9-2-1 Time-Varying Fields in Stationary Circuits

Faraday's Law is the basis for electric power **generators**!

HO: The Electromotive Force

Faraday's Law is likewise the basis for the operation of **transformers**!

HO: The Ideal Transformer

HO: Eddy Currents

Faraday's Law of Induction

Say instead of a static magnetic flux density, we consider a **time-varying** B field (i.e., $\mathbf{B}(\bar{r},t)$). Recall that one of **Maxwell's** equations is:

$$\nabla \times \mathbf{E}(\bar{r}) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

Yikes! The curl of the electric field is therefore **not zero** if the magnetic flux density is **time-varying**!

If the magnetic flux density is changing with time, the electric field will **not be conservative**!

Q: *What the heck does this equation mean ?!?*

A: Integrate both sides over some surface S:

$$\iint_S \nabla \times \mathbf{E}(\bar{r}) \cdot \overline{ds} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$

Applying **Stoke's Theorem**, we get:

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$

where C is the contour that surrounds the boundary of S.

Note that $\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} \neq 0$.

This equation is called **Faraday's Law of Induction**.

Q: *Again, what does this **mean**?*

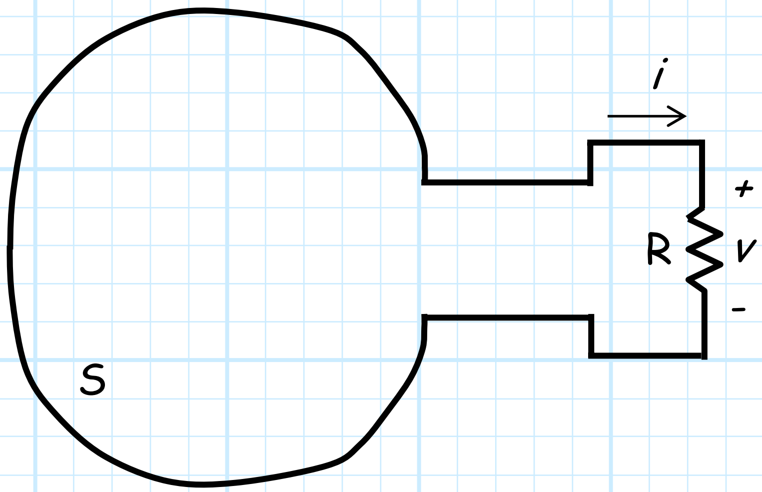
A: It means that a time varying magnetic flux density $\mathbf{B}(\bar{r},t)$ can **induce** an electric field (and thus an electric potential difference)!

Faraday's Law describes the behavior of such devices such as **generators**, **inductors**, and **transformers** !

Michael Faraday (1791-1867), an English chemist and physicist, is shown here in an early daguerreotype holding a bar of glass he used in his 1845 experiments on the effects of a magnetic field on polarized light. Faraday is considered by many scientists to be the greatest experimentalist ever! (from "Famous Physicists and Astronomers" www.phy.hr/~dpaar/fizicari/index.html)

The Electromotive Force

Consider a wire loop with surface area S, connected to a single **resistor** R.

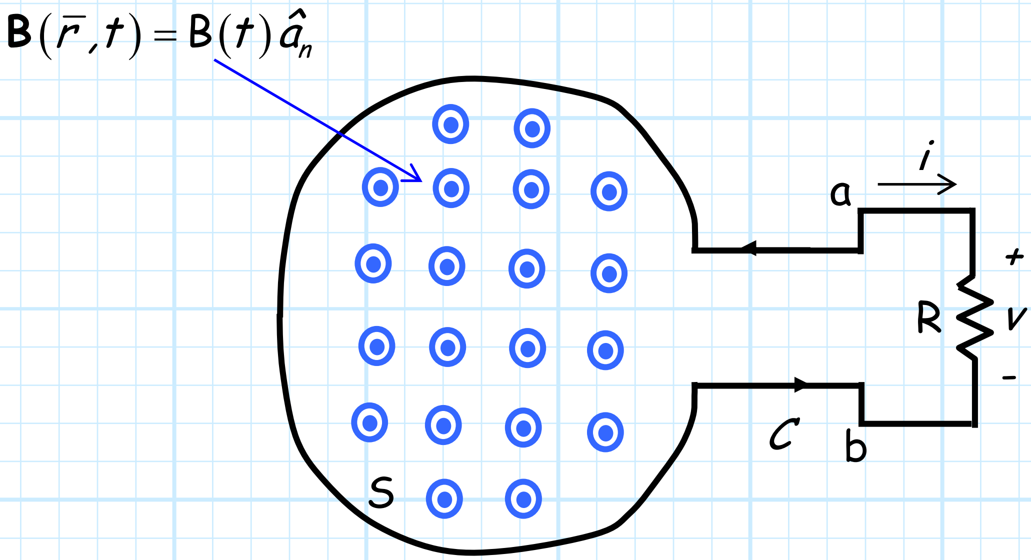


Since there is **no** voltage or current **source** in this circuit, both voltage v and current i are **zero**.

Now consider the case where there is a **time-varying** magnetic flux density $\mathbf{B}(\bar{r},t)$ within the loop only. In other words, the magnetic flux density outside the loop is zero (i.e., $\mathbf{B}(\bar{r},t)=0$ outside of S).

Say that this magnetic flux density is a **constant** with respect to position, and points in the direction normal to the surface S. In other words;

$$\mathbf{B}(\bar{r},t)=B(t)\ \hat{a}_n$$



According to **Faraday's Law**:

$$\oint_C \mathbf{E}(\bar{r})\cdot\overline{d\ell} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}(\bar{r},t)\cdot\overline{ds}$$

$$\oint_C \mathbf{E}(\bar{r})\cdot\overline{d\ell} = -\frac{\partial}{\partial t}\iint_S B(t)\,\hat{a}_n\cdot\overline{ds}$$

$$\int_a^b \mathbf{E}(\bar{r})\cdot\overline{d\ell} + \int_b^a \mathbf{E}(\bar{r})\cdot\overline{d\ell} = -\frac{\partial B(t)}{\partial t}\iint_S ds$$

The contour from point a to point b is along a wire, which we presume to be a **perfect conductor**. Since the electric field within a perfect conductor is equal to **zero**, we find:

$$\int_a^b \mathbf{E}(\bar{r})\cdot\overline{d\ell} = 0$$

Likewise, if we integrate through the **resistor** from point *b* to point *a,* we find:

$$\int_b^a \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -v$$

Finally, we note that:

$$\iint_S ds = S$$

where S is the **surface area** of the loop.

Combining these results, we find:

$$v = S \frac{\partial B(t)}{\partial t}$$

Or, recalling that **magnetic flux** Φ is defined as:

$$\iint_S \mathbf{B}(\bar{r},t) \cdot \overline{ds} = \Phi(t)$$

we can write:

$$v = S \frac{\partial B(t)}{\partial t} = \frac{\partial \Phi(t)}{\partial t}$$

For this case, the **voltage** across the resistor is proportional to the **time derivative** of the **total magnetic flux** passing through the aperture formed by contour *C*.

Using the circuit form of Ohm's Law, we likewise find that the current in the circuit is:

$$
\begin{aligned}
i &= \frac{v}{R} \\
&= \frac{S}{R}\frac{\partial B(t)}{\partial t} \\
&= \frac{1}{R}\frac{\partial \Phi(t)}{\partial t}
\end{aligned}
$$

In other words, time-varying magnetic flux density can **induce** a voltage and current in a circuit, even though there are **no** voltage or current **sources** present!

The voltage created is known as the **electromotive force.**

The electromotive force is the basic **phenomenon** behind the behavior of:

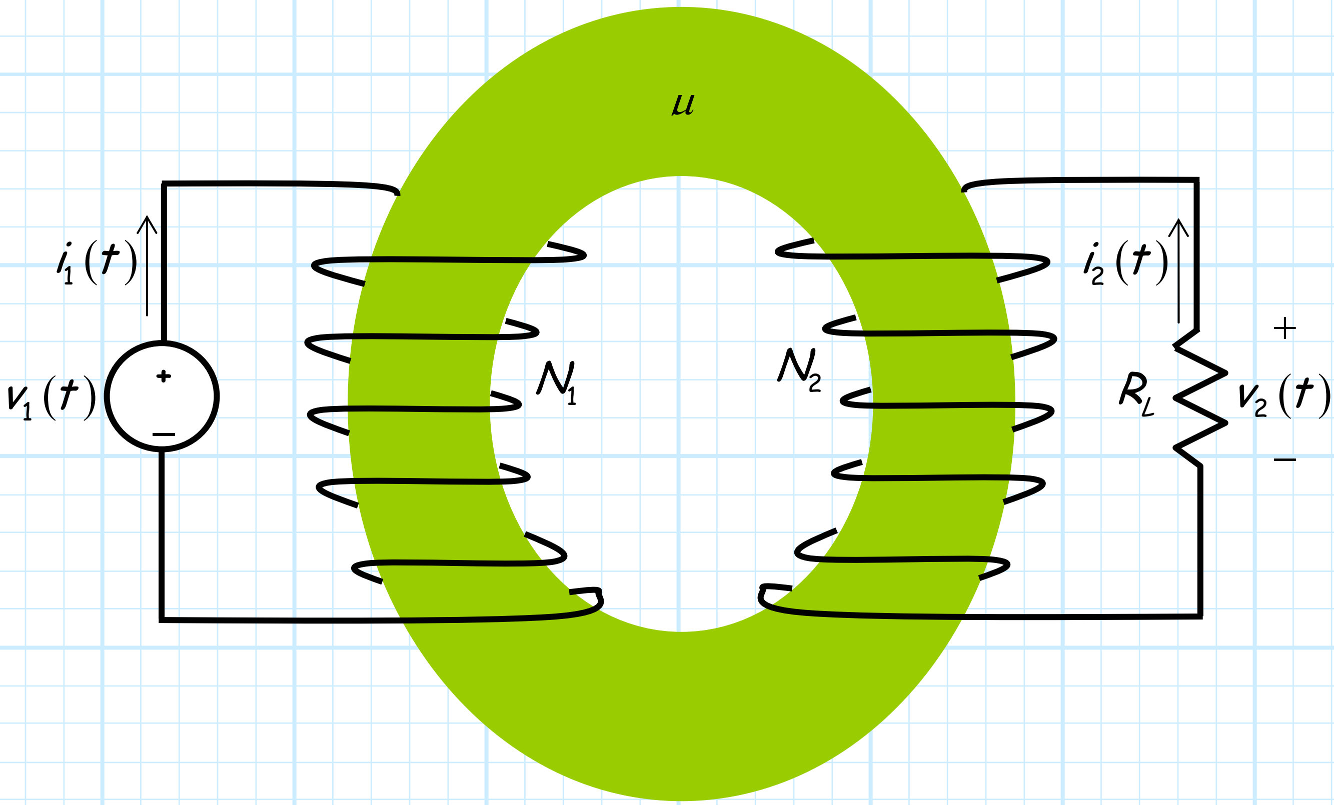
1. Electric power **generators**

2. Transformers

3. Inductors

The Ideal Transformer

Consider the structure:



* The "doughnut" is a ring made of **magnetic material** with **very** large relative permeability (i.e., $\mu_r >> 1$).

* On one side of the ring is a coil of wire with N_1 turns. This could of wire forms a **solenoid**!

* On the other side of the ring is **another** solenoid, consisting of a coil of N_2 turns.

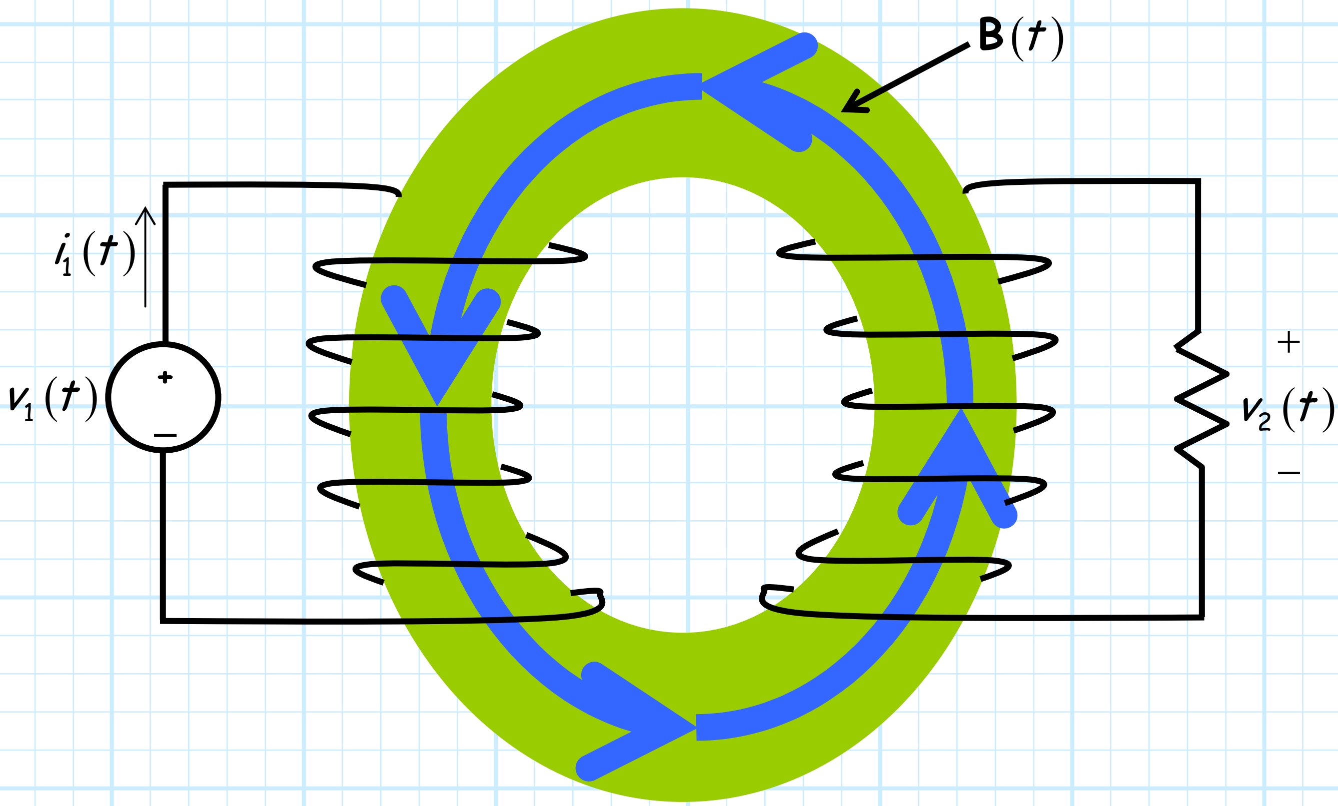
This structure is an **ideal transformer** !

* The solenoid on the left is the **primary loop**, where the one on the right is called the **secondary loop**.

The current $i_1(t)$ in the primary generates a **magnetic flux density** $\mathbf{B}(\bar{r},t)$. Recall for a solenoid, this flux density is approximately **constant** across the solenoid cross-section (i.e., with respect to $\bar{r}$). Therefore, we find that the magnetic flux density within the solenoid can be written as:

$$\mathbf{B}(\bar{r},t) = \mathbf{B}(t)$$

It turns out, since the permeability of the ring is **very large**, then this flux density will be **contained** almost entirely **within** the magnetic ring.



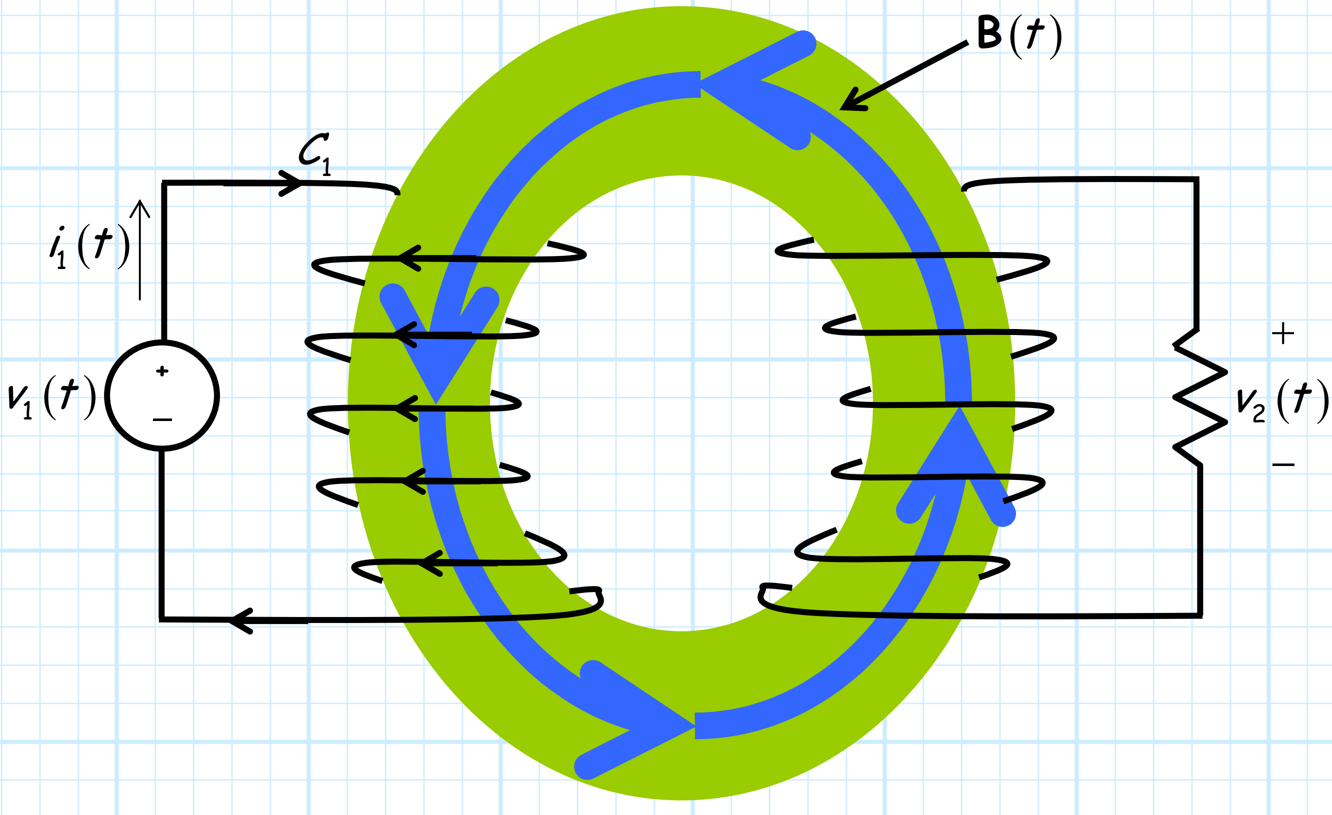
Therefore, we find that the magnetic flux density in the **secondary** solenoid is **equal** to that produced in the **primary**!

Q: *Does this mean also that $v_1(t) = v_2(t)$?*

A: Let's apply **Faraday's Law** and find out!

Applying Faraday's Law to the **primary** loop, defined as **contour** C_1, we get:

$$\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint_{S_1} \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$



Q: *But, **contour** C_1 follows the wire of the **solenoid**. What the heck then is **surface** S_1??*

A: S_1 is the surface of a **spiral**!

We can approximate the **surface area** of a spiral by first considering the surface area formed by a **single loop** of wire, denoted S_0. The surface area of a spiral of N **turns** is therefore approximately N S_0. Thus, we say:

$$\iint_{S_1} \mathbf{B}(t) \cdot \overline{ds} = N_1 \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds}$$

Likewise, we find that by integrating around **contour** C_1:

$$-\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v_1(t)$$

Faraday's Law therefore becomes:

$$v_1(t) = N_1 \frac{\partial}{\partial t} \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds}$$

$$= N_1 \frac{\partial \Phi(t)}{\partial t}$$

where $\Phi(t)$ is the total **magnetic flux** flowing through the solenoid:

$$\Phi(t) = \iint_{S_0} \mathbf{B}(t) \cdot \overline{ds}$$

Remember, this **same** magnetic flux is flowing through the **secondary** solenoid as well. Faraday's Law for this solenoid is:

$$\oint_{C_2} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint_{S_2} \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$

where we similarly find that:

$$-\oint_{C_2} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v_2(t)$$

and:

$$\frac{\partial}{\partial t} \iint_{S_2} \mathbf{B}(\bar{r},t) \cdot \overline{ds} = N_2 \frac{\partial}{\partial t} \iint_{S_0} \mathbf{B}(\bar{r},t) \cdot \overline{ds}$$
$$= N_2 \frac{\partial \Phi(t)}{\partial t}$$

therefore we find that :

$$v_2(t) = N_2 \frac{\partial \Phi(t)}{\partial t}$$

Combining this with our expression for the primary, we get:

$$\frac{\partial \Phi(t)}{\partial t} = \frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$

As a result, we find that the voltage $v_2(t)$ across the **load resistor** R_L is related to the voltage **source** $v_1(t)$ as:

$$v_2(t) = \frac{N_2}{N_1} v_1(t)$$

Note that by changing the number of the **ratio** of windings N in each solenoid, a transformer can be constructed such that the output voltage $v_2(t)$ is either much **greater** than the input voltage $v_1(t)$(i.e., $N_2/N_1 >> 1$), or much **less** than the input voltage (i.e., $N_2/N_1 << 1$).

We call the first case a **step-up** transformer, and the later case a **step-down** transformer.

Q: *How are the currents $i_1(t)$ and $i_2(t)$ related ??*

A: Energy must be **conserved**!

Since a transformer is a **passive** device, it cannot **create** energy. We can state therefore that the power **absorbed** by the resistor must be equal to the power **delivered** by the voltage source.

In other words:

$$Power = v_1(t)\, i_1(t) = -v_2(t)\, i_2(t)$$

The minus sign in the above expression comes from the definition of $i_2(t)$, which is pointing **into** the transformer (as opposed to pointing into the resistor).

Rearranging the above expression, we find:

$$\begin{aligned} i_2(t) &= -\frac{v_1(t)}{v_2(t)}\, i_1(t) \\ &= -\frac{N_1}{N_2}\, i_1(t) \end{aligned}$$

Note that for a **step-up** transformer, the output current $i_2(t)$ is actually **less** than that of $i_1(t)$, whereas for the step-down transformer the opposite is true.

Thus, if the **voltage is increased**, the **current is decreased** proportionally—energy is conserved!

Finally, we note that the primary of the transformer has the apparent **resistance** of:

$$R_1 \doteq \frac{v_1}{i_1} = \frac{v_1}{v_2}\frac{v_2}{i_2}\frac{i_2}{i_1} = \frac{N_1}{N_2}\left(-R_L\right)\frac{-N_1}{N_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Thus, we find that for a **step-up** transformer, the primary resistance is much **greater** than that of the load resistance on the secondary. Conversely, a **step-down** transformer will exhibit a primary resistance R_1 that is much **smaller** than that of the load.

One more **important** note! We applied conservation of energy to this problem because a transformer is a passive device. Unlike an active device (e.g., current or voltage source) it cannot **add** energy to the system .

However, passive devices can certainly **extract** energy from the system!

Q: *How can they do this?*

A: They can convert electromagnetic energy to **heat** !

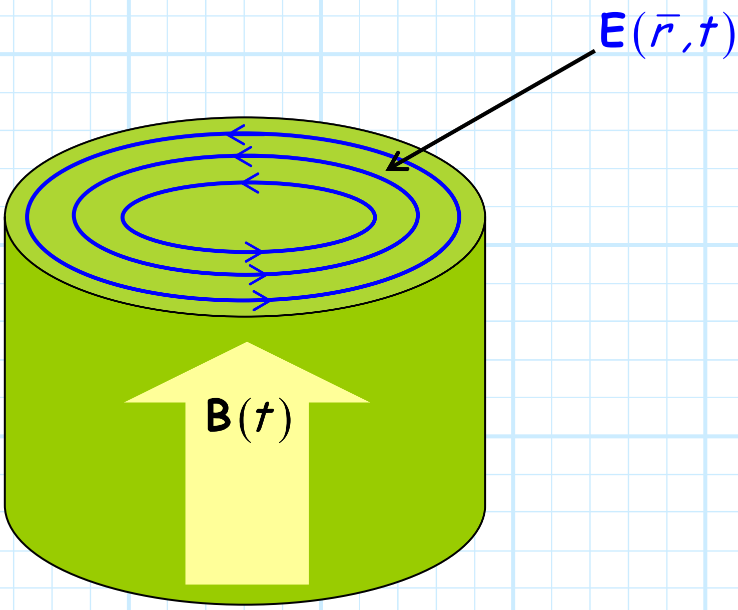
If the "doughnut" is lossy (i.e., **conductive**), electric **currents** $\mathbf{J}(\bar{r})$ can be **induced** in the magnetic material. The result are **ohmic losses**, which is power delivered to some volume V (e.g., the doughnut) and then converted to **heat**. This loss can be determined from **Joule's Law**:

$$P_{loss} = \iiint_V \sigma \left|\mathbf{E}(\bar{r})\right|^2 dv \qquad [W]$$

In this case, the transformer is **non-ideal**, and the expressions derived in this handout are only **approximate**.

Eddy Currents

From **Faraday's Law**, we know that a **time-varying** magnetic flux density $\mathbf{B}(t)$ will **induce** electric fields $\mathbf{E}(\bar{r},t)$. Consider what happens if this time-varying magnetic flux density occurs **within** some **material**, say the **magnetic core** of some solenoid.

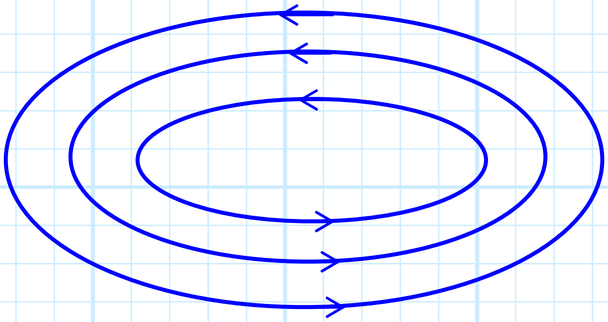


If the material is **non-conducting** (i.e., $\sigma = 0$), then these induced electric fields essentially cause **no** problems. But consider what happens if the material **is** conducting. In this case, we apply Ohm's Law and find that **current** $\mathbf{J}(\bar{r})$ is the result:

$$\mathbf{J}(\bar{r}) = \sigma(\bar{r})\,\mathbf{E}(\bar{r})$$

We find that these currents **swirl** around in the media in a **solenoidal** manner (i.e., $\nabla \cdot \mathbf{J}(\bar{r}) = 0$ and $\nabla \times \mathbf{J}(\bar{r}) \neq 0$).

We call these currents **Eddy Currents.**



Eddy currents are **problematic** in the magnetic cores of transformers, generators, and inductors, as they result in **Ohmic Losses**. These losses in power can be determined from Joules Law as:

$$P_{loss} = \iiint_V \mathbf{E}(\bar{r}) \cdot \mathbf{J}(\bar{r})\, dv$$
$$= \iiint_V \sigma \left|\mathbf{E}(\bar{r})\right|^2 dv \qquad [W]$$

where V is the volume of the magnetic core. The "lost" power is of course simply transferred to **heat**.

It is evident that if conductivity is **low** (i.e., $\sigma \approx 0$), the eddy currents and their resulting losses will be **small**. Ideally, then, we seek a magnetic material that has **very high** permeability and **very low** conductivity.

Oh, it also should be **inexpensive**!

Finding a material with these three attributes is **very** difficult!