9-3 Inductance

Reading Assignment: *pp. 290-286*

* A transformer is an example of **mutual inductance**, where a time-varying current in **one** circuit (i.e., the primary) induces an emf voltage in **another** circuit (i.e., the secondary).

* We likewise can have **self inductance**, were a time-varying current in a circuit induces an emf voltage within that **same** circuit!

In fact, we can create circuit structures where this induced emf will be **very large**—we call these circuit elements **inductors**!

Q: *So how do we **make** an inductor?*

A: Typically, an inductor is a **solenoid**!

HO: Inductance

Example: The Inductance of a Solenoid

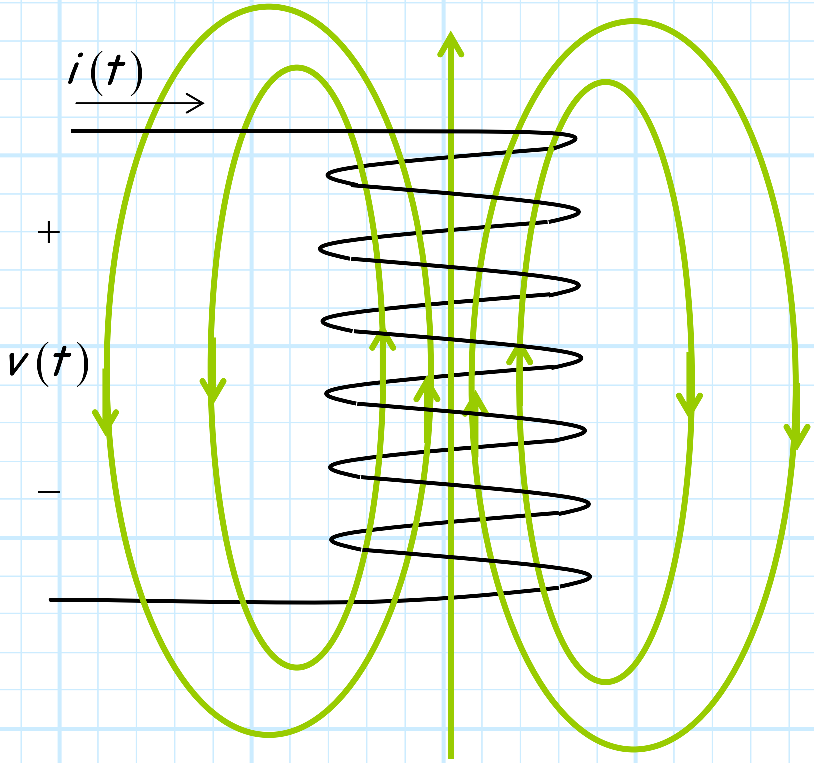
Example: The Inductance of a Coaxial Transmission Line

Just like a capacitor, an inductor can **store energy**!

HO: Energy and Magnetic Fields

Inductance

Consider a **solenoid** with N turns:



The current $i(t)$ in flowing in the wire will produce a time-varying magnetic flux density within the solenoid. This time-varying magnetic flux density will **induce a voltage** $v(t)$ across the solenoid.

This voltage can be determined using **Faraday's Law**:

$$-\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = \frac{\partial}{\partial t} \iint_{S_1} \mathbf{B}(\bar{r}) \cdot \overline{ds}$$

Just like we determined for the **ideal transformer**, we find that:

$$-\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v(t)$$

and that:

$$\frac{\partial}{\partial t}\iint_{S_1} \mathbf{B}(\bar{r}) \cdot \overline{ds} = \frac{\partial}{\partial t} N \iint_{S_0} \mathbf{B}(\bar{r}) \cdot \overline{ds}$$

$$= N\frac{\partial \Phi(t)}{\partial t}$$

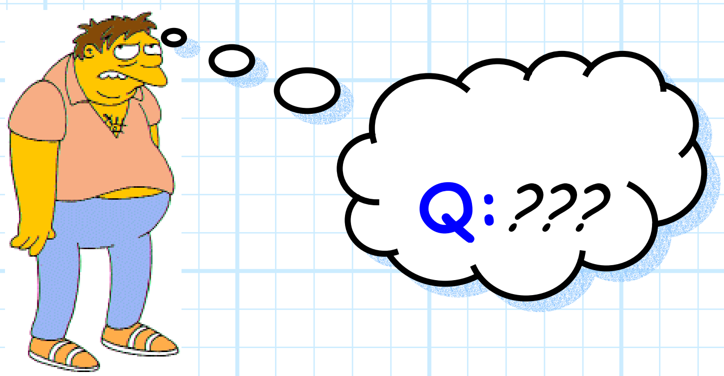
where S_0 is the surface area of **one** loop.

Therefore, just as we determined for a transformer, Faraday's Law says that:

$$v(t) = N\frac{\partial \Phi(t)}{\partial t}$$

Now, let's **define** the product $N\,\Phi(t)$ as:

$$N\,\Phi(t) \doteq \Lambda(t) = \text{flux linkages} \quad [\text{Webers}]$$



A: A magnetic flux of $\Phi(t)$ Webers passes through **each and every** one of the N loops of the solenoid. We say therefore that each loop surrounds, or "**links**" $\Phi(t)$ Webers of flux. If there are *N* loops, then the solenoid links a **total** of $N\,\Phi(t)$ Webers of flux. We call therefore $N\,\Phi(t)$ the total **flux linkages** surrounded by the solenoid.

Thus we can state our induced solenoid voltage as the time derivative of the flux linked by the solenoid:

$$v(t) = \frac{\partial \Lambda(t)}{\partial t}$$

Now, recall that current *i(t)* produced the magnetic flux density and thus the magnetic flux. As a result, we find that the current *i(t)* is **directly proportional** to the total flux linkages of the solenoid:

$$i(t) \propto \Lambda(t)$$

Lets define the **proportionality constant** as *L*, so that we can say:

$$\Lambda(t) = L\ i(t)$$

Since *i(t)* has units of amps and $\Lambda(t)$ the units of Webers, the constant L must have units of **Webers/Amp**.

Taking the **time derivative** we thus find:

$$\frac{\partial \Lambda(t)}{\partial t} = L \frac{\partial i(t)}{\partial t}$$

Note we can now write the **induced voltage** as:

$$v(t) = L \frac{\partial i(t)}{\partial t}$$

Q: Look familiar?

A: *Of course, L is **inductance**!*

Inductance is therefore defined as the **ratio** of current i to the total flux linkages it creates!

$$L \doteq \frac{\Lambda}{i} = \text{inductance} \quad \left[\frac{\text{Webers}}{\text{Amp}}\right]$$

Inductance is obviously dependent on the **structure** of the device (e.g., number of loops, diameter, length).

By the way, we have another name for Webers/Amp—**Henries**!

$$\text{Henries} \doteq \frac{\text{Webers}}{\text{Ampere}}$$

Example: The Inductance of a Solenoid

Many **inductors** used in electronic circuits are simply **solenoids**. Let's determine the **inductance** of this structure!

First, we recall that inductance is the **ratio** of the **current** and the **flux linkages** that the current produces:

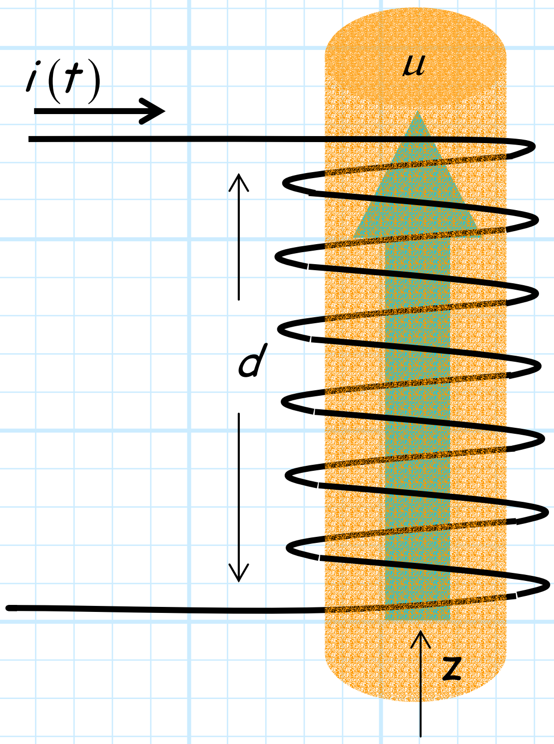
$$L \doteq \frac{\Lambda}{i} = \text{inductance} \quad \left[\frac{\text{Webers}}{\text{Amp}}\right]$$

The question then is, what is flux linkages Λ for a **solenoid**?

Recall that the magnetic flux density in the **interior** of a solenoid is:

$$\mathbf{B}(\bar{r}) \approx \frac{\mu N i}{d} \hat{a}_z$$

where N is the number of loops and d is the length of the solenoid.



The total **magnetic flux** flowing through the solenoid is therefore found by integrating across the **cross-section** of the solenoid:

$$\begin{aligned}\Phi &= \iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} \\ &= \frac{\mu N i}{d} S\end{aligned}$$

where S is the cross-sectional area of the solenoid (e.g., $S = \pi a^2$ if solenoid is circular with radius a).

Recall the total **flux linkage** is just the **product** of the **magnetic flux** and the **number of loops**:

$$\begin{aligned}\Lambda &= N\Phi \\ &= \frac{\mu N^2 S}{d} i\end{aligned}$$

Thus, we now find that the **inductance of a solenoid** is:

$$L = \frac{\Lambda}{i} = \frac{\mu N^2 S}{d}$$

Note if we wish to **increase** the inductance of this solenoid, we can either:

1) **Increase** the **permeability** μ of the core material.

2) **Increase** the number of **turns** N.

3) **Increase** the cross-sectional **area** S

4) **Decrease** the **length** d (while keeping N constant).

Note all of the derivations in this handout are derived from the solution to an **infinite** solenoid. As a result, they are **approximations**, but are typically accurate ones **provided** that:

$$d >> \sqrt{S}$$

In other words, provided that the inductor **length** is significantly **greater** than its **radius**.

The Inductance of a Coaxial Transmission Line

Recall that we earlier determined the **capacitance** (per unit length) of a **coaxial transmission line** to be:

$$\frac{C}{\ell} = \frac{2\pi\,\varepsilon}{\ln[b/a]} \qquad \left[\frac{\text{farads}}{\text{meter}}\right]$$

We can likewise determine its **inductance** per unit length.

Q: *Yikes! How do we accomplish this? There are no **loops** in a coaxial line!*

A: True. We instead begin by determining the **energy stored** (per unit length) of a coax line.

Recall that the magnetic flux density **between** the inner and outer conductors of a coaxial line is:

$$\mathbf{B}(\bar{r}) = \frac{\mu I}{2\pi\rho}\,\hat{a}_\phi \qquad (a < \rho < b)$$

Therefore the **magnetic field** within the line is:

$$\mathbf{H}(\bar{r}) = \frac{I}{2\pi\rho}\,\hat{a}_\phi \qquad (a < \rho < b)$$

The **energy stored** in a length ℓ of the coax line is therefore:

$$\begin{aligned} W_m &= \frac{1}{2}\iiint \mathbf{B}\cdot\mathbf{H}\,dv \\ &= \frac{\mu I^2}{8\pi^2}\int_0^{\ell}\int_a^{b}\int_0^{2\pi}\frac{1}{\rho^2}\,\hat{a}_\phi\cdot\hat{a}_\phi\,\rho\,d\rho\,d\phi\,dz \\ &= \frac{\mu I^2 \ell}{4\pi}\ln\left[\frac{b}{a}\right] \end{aligned}$$

Q: *So what? We want to find the **inductance** of the line, **not** the energy stored in it!*

A: True. But recall inductance is **related** to stored energy as:

$$W_m = \frac{1}{2}LI^2$$

Or in other words:

$$L = \frac{2W_m}{I^2}$$

Using this expression, we find:

$$\begin{aligned} L &= \frac{2}{I^2}\left(\frac{\mu I^2\,\ell}{4\pi}\ln\left[\frac{b}{a}\right]\right) \\ &= \frac{\mu}{2\pi}\ln\left[\frac{b}{a}\right]\ell \end{aligned}$$

Or, in other words, the **inductance per unit length** of a coax transmission line is:

$$\frac{L}{\ell} = \frac{\mu}{2\pi} \ln\left[\frac{b}{a}\right] \qquad \left[\frac{\text{Henries}}{\text{m}}\right]$$

Note here that we did **not** consider the magnetic fields **within the conductors**. For most engineering applications (i.e., **time-varying**), we will find that the contribution of these fields are small and thus can be **neglected**.

Energy and Magnetic Fields

Recall that the **energy stored** in an **electro**static system is:

$$W_e = \frac{1}{2}\iiint_V \rho_v(\bar{r})\, V(\bar{r})\, dv$$

or equivalently:

$$W_e = \frac{1}{2}\iiint_V \mathbf{D}(\bar{r}) \cdot \mathbf{E}(\bar{r})\, dv$$

This led to the expression relating energy and **capacitance**:

$$W_e = \frac{1}{2} C V^2$$

We can similarly ask the question, how much **energy** is stored in a **magneto**static system?

Precisely the amount of **work** required to establish the **current density** $\mathbf{J}(\bar{r})$!

We find that the expressions for this work/energy are **analogous** to that of electrostatics. For example, we find that:

$$W_m = \frac{1}{2}\iiint_V \mathbf{J}(\bar{r})\cdot\mathbf{A}(\bar{r})\,dv$$

Therefore, we **again** find that energy stored is equal to the integration of the "product" of the **sources** (e.g., ρ_v or $\mathbf{J}$) and the **potential** function (e.g., V or $\mathbf{A}$).

Likewise, this energy can be expressed in terms of the two magnetic **fields**:

$$W_m = \frac{1}{2}\iiint_V \mathbf{B}(\bar{r})\cdot\mathbf{H}(\bar{r})\,dv$$

Therefore, we again find that energy stored is equal to the integration of the dot product of the **flux density** (e.g., $\mathbf{D}$ or $\mathbf{B}$) and the other **field** (e.g., $\mathbf{E}$ or $\mathbf{H}$).

We likewise find that this energy can be directly expressed for the energy stored by an **inductor**:

$$W_m = \frac{1}{2}LI^2$$

Look **familiar** ?