# 3-3 Current and Current Density

**Reading Assignment:** ***pp.63-68***

Charge is often **moving**!

→

HO: Charge and Current

## A. Volume Current Density

**Problem!:** Most often, charge is **not** restricted to a wire, but instead flows "willy nilly" throughout some volume V.

**Q:**

**A:** HO: Volume Current Density

**Q:**

**A:** HO: The Current $I$ through Surface S

## B. Surface Current Density

HO: Surface Current Density

## C. Charge Velocity

If charges are moving, then they must have some **velocity**.

**Q:**

**A:** HO: Charge Velocity and Current Density

# Charge and Current

Say we have a conductor (e.g., wire) with $I$=1 Ampere of current flowing through it.


**Q:** *What does this mean,* ***physically*** *?*

**A:** Current $I$ simply describes the **rate** at which **net** charge passes through the wire cross-sectional surface S. For example, if a **net** charge $\Delta Q$ moves across surface S in some small amount of time $\Delta t$, we find that:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

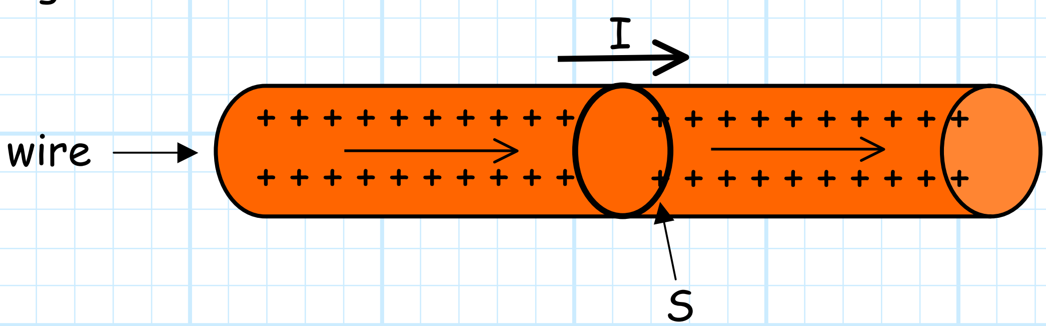
Thus, we find that 1 Amp means **+1.0 Coulomb** of net charge passes by a location on the wire each **second**, with the net charge in this case flowing from left to right.

**Q:** *The current is **positive**, does this mean that the current is made up of **positive** charge?*

**A:** **No!** Current generally consists of **both** positively and negatively charged particles.

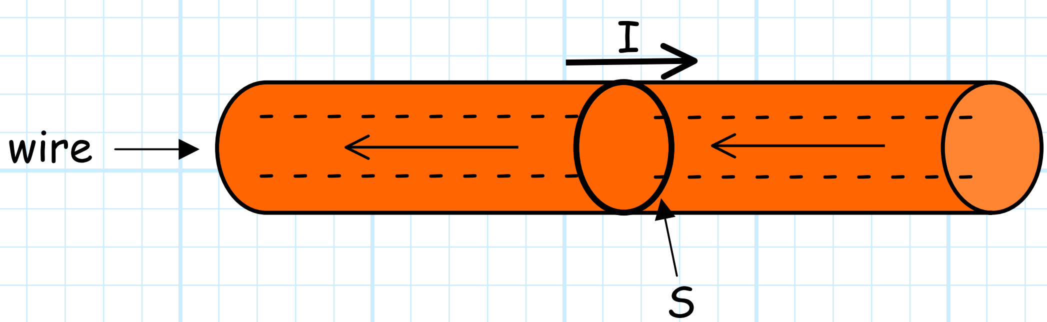
Remember, current is the **net** change in charge with respect to time.

For example, say **positive** charges are moving from **left to right** through the wire:




The current due to these charges is **positive**, as the total net charge on the right side of the surface is **increasing** with time.

That was pretty obvious, but here's the **tricky** part: say **negative** charges are moving from **right to left** through the wire (the **opposite** direction of that above).



Note in this case, the total charge on the right side of S is **again increasing** !

* With the first case, the net charge was increasing because positive charges were entering the right side. For this case, the net charge on the right side is **also** increasing, but because negative charge is **leaving** the right side !

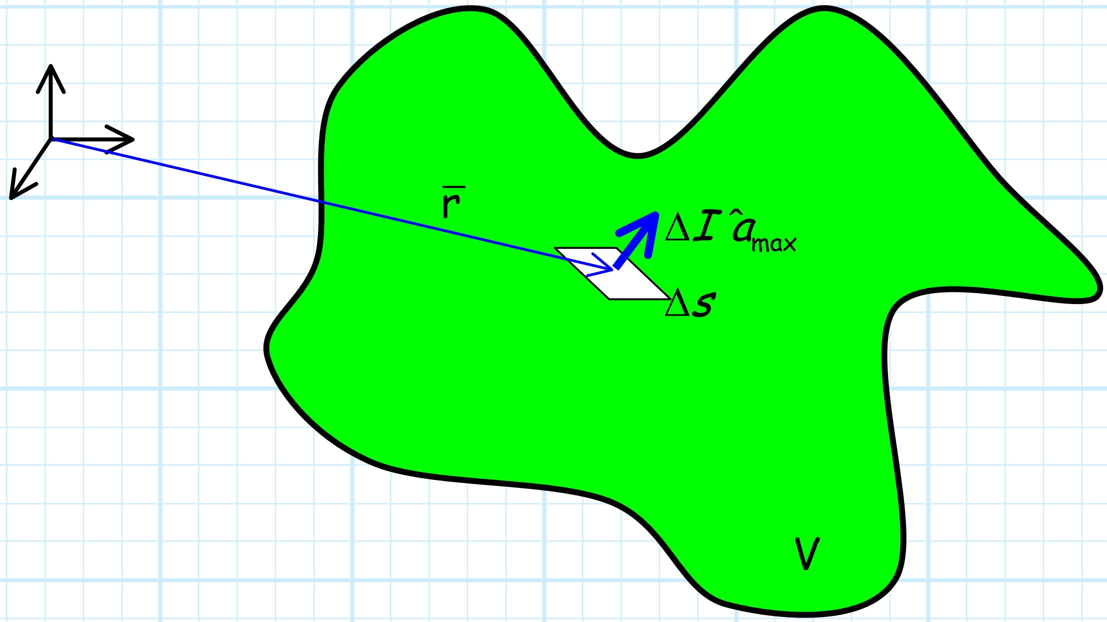
* For reasons we shall learn about later, if positive charge moves one direction, then negative charge will generally move in the **opposite** direction. Therefore, total current is composed of charges moving in **both** directions:

$$I = I^{+} + I^{-}$$

* Generally speaking, it **does not matter** (in fact we generally cannot tell) whether the particles that form a specific current are negative or positive—all that matters is the **net** change in charge across a surface.

# Volume Current Density

Say at a given point $\bar{r}$ located in a volume V, charge is moving in **direction** $\hat{a}_{max}$.




Now, consider a **small surface** $\Delta s$ that is centered at the point denoted by $\bar{r}$, and oriented such that it is orthogonal to unit vector $\hat{a}_{max}$. Since charge is moving across this small surface at some rate (coulombs/sec), we can define a **current** $\Delta I = \Delta Q / \Delta t$ that represents the current flowing through $\Delta s$.

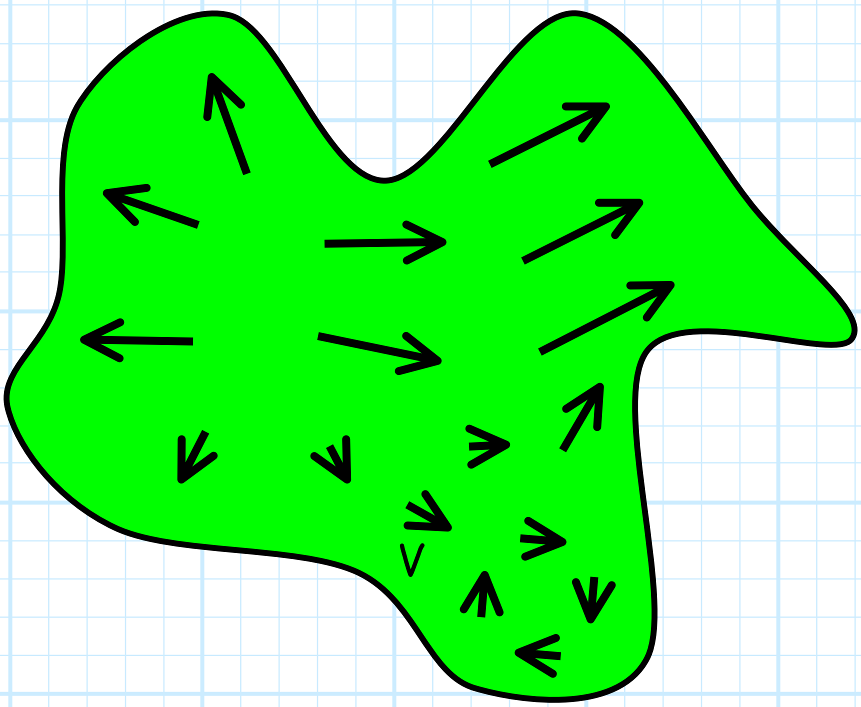
Note **vector** $\Delta I \, \hat{a}_{max}$ therefore represents both the **magnitude** $(\Delta I)$ and **direction** $\hat{a}_{max}$ of the current flowing through surface area $\Delta s$ at point $\bar{r}$.

From this, we can define a **volume current density** $\mathbf{J}(\bar{r})$ at each and every point $\bar{r}$ in volume V by **normalizing** $\Delta I\ \hat{a}_{max}$ by dividing by the surface area $\Delta s$:

$$\mathbf{J}(\bar{r}) = \lim_{\Delta s \to 0} \frac{\Delta I\ \hat{a}_{max}}{\Delta s} \qquad \left[\frac{\text{Amps}}{\text{m}^2}\right]$$

The result is a **vector** field !

For example, current density $\mathbf{J}(\bar{r})$ might look like:




**NOTE:** The **unit** of **volume** current density is **current/area**; for example, $A/m^2$.

# The Current I through Surface S

Given that we know volume current density $\mathbf{J}(\bar{r})$ throughout some volume, we can find the **total current** through **any** arbitrary **surface S** as:

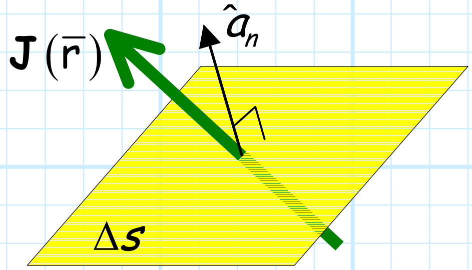
$$I = \iint_S \mathbf{J}(\bar{r}_s) \cdot \overline{ds} \qquad [Amps]$$

This integral is in the form of the **surface integral** we studied in Section 2-5.

Note the **integrand** has units of **current** (amps):

$$\mathbf{J}(\bar{r}_s) \cdot \overline{ds} = J_n(\bar{r}_s)\left|\overline{ds}\right| \qquad \left[\left(\frac{\text{Amps}}{\text{m}^2}\right)\left(\text{m}^2\right) = \text{Amps}\right]$$

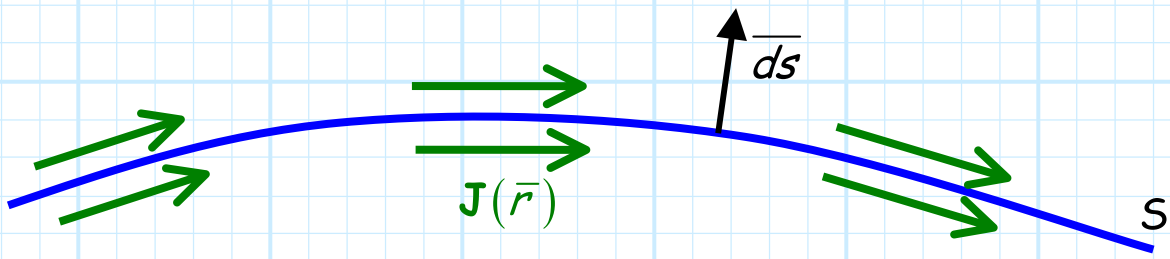
Physically, the value $\Delta I = \mathbf{J}(\bar{r}) \cdot \overline{ds}$ is the current flowing **through** the tiny differential surface $\Delta s$, located at point $\bar{r}$ on surface $S$.



* Therefore if we **add** up (i.e., integrate) the current flowing through **each** and every differential surface element $\Delta s$ that makes up surface $S$, we determine the **total** current $I$ flowing **through** surface $S$.

* Note the **sign** of current $I$ is determined by the **direction** of differential surface vector $\overline{ds}$. For **example**, if $I$ is **positive**, then the current is flowing **through** the surface in the direction of $\overline{ds}$.

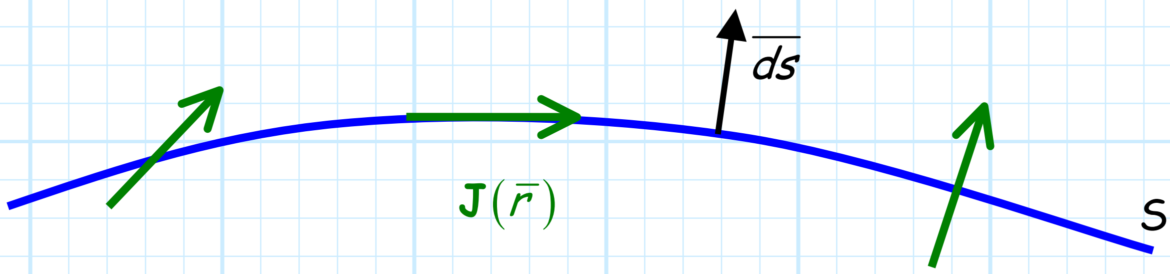
* So, consider the case where $\mathbf{J}(\bar{r})$ describes current that is flowing **tangential** to **every point** on surface S. In other words, the current density has no **normal** component on the surface S !




As a result, we find that $\mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$ at **every** point on the surface, and therefore the surface **integral** results in $I = 0$.

This of course is **physically** the correct answer! Current is flowing **along** the surface, but none is flowing **through** it.

To get a **non-zero** amount of total current, the current density must have a **normal** component at **some** points on the surface.




For the case above, $I \neq 0$.

**Q:** *We know that if* $\mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$ *at all points on a surface, then the current flowing through the surface is zero (I=0).*

*Is the* ***converse*** *true? That is,* ***if*** *the total current through a surface is* ***zero****, does that mean that the current density is* ***tangential*** *to the surface at* ***all*** *points?*

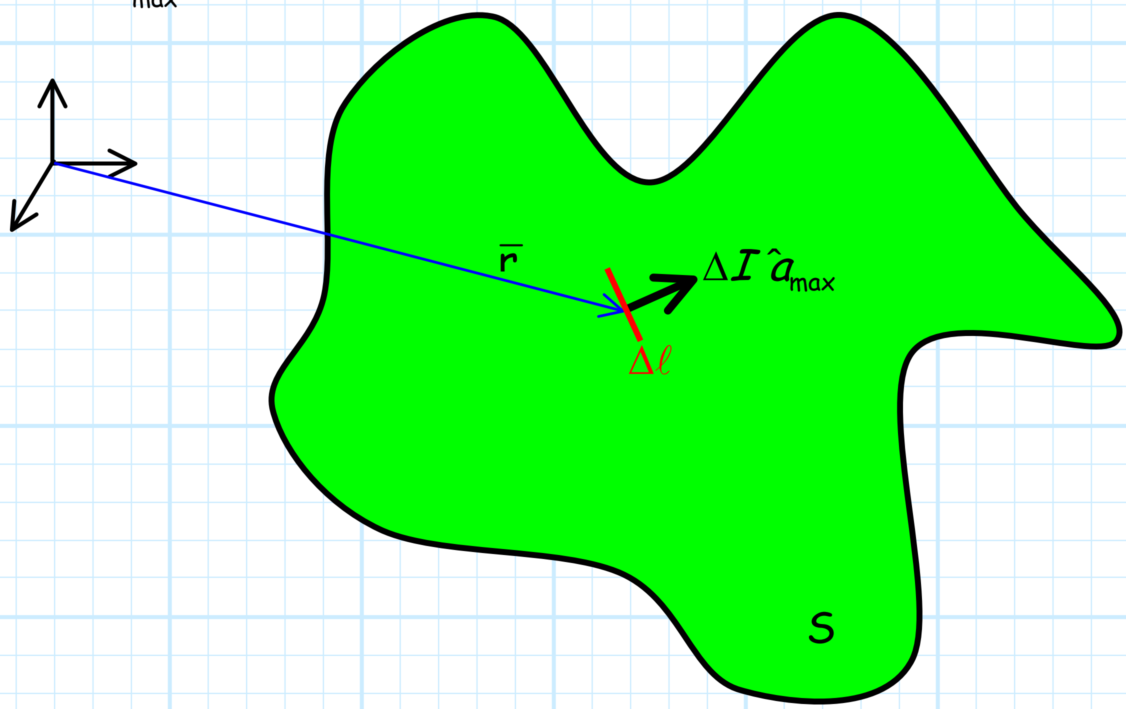
**A:**

# Surface Current Density

Consider now the problem where we have moving **surface** charge $\rho_s(\bar{r})$.

➡ The result is **surface** current!

Say at a given point $\bar{r}$ located on a surface $S$, charge is moving in **direction** $\hat{a}_{max}$.




Now, consider a **small length** of contour $\Delta\ell$ that is centered at point $\bar{r}$, and oriented such that it is orthogonal to unit vector $\hat{a}_{max}$. Since charge is moving across this small length, we can define a **current** $\Delta I$ that represents the current flowing across $\Delta\ell$.

Note **vector** $\Delta I \, \hat{a}_{max}$ therefore represents both the **magnitude** $(\Delta I)$ and **direction** $\hat{a}_{max}$ of the current flowing across contour $\Delta \ell$ at point $\bar{r}$.

From this, we can define a **surface current density** $\mathbf{J}_s(\bar{r})$ at every point $\bar{r}$ on surface S by **normalizing** $\Delta I \, \hat{a}_{max}$ by dividing by the length $\Delta \ell$:

$$\mathbf{J}_s(\bar{r}) = \lim_{\Delta \ell \rightarrow 0} \frac{\Delta I \, \hat{a}_{max}}{\Delta \ell} \qquad \left[\frac{\text{Amps}}{\text{m}}\right]$$

The result is a **vector** field !

**NOTE:** *The unit of **surface** current density is current/**length**; for example, A/m.*

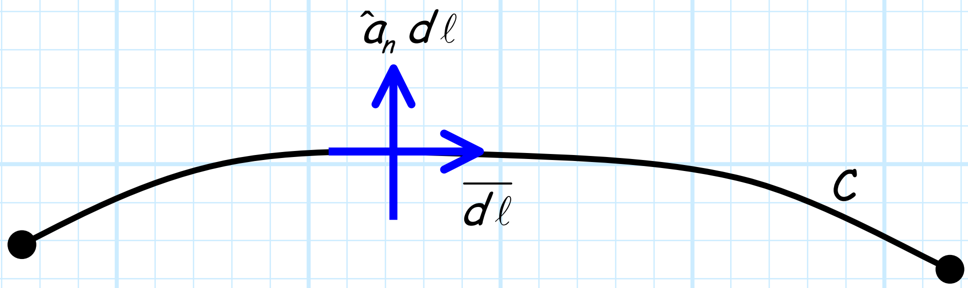
Given that we know surface current density $\mathbf{J}_s(\bar{r})$ throughout some volume, we can find the total **current** across **any** arbitrary **contour** ***C*** as:

$$I = \int_C \mathbf{J}_s(\bar{r}) \cdot \hat{a}_n \, d\ell$$

This looks very much like the contour integral we studied in the previous chapter. However, there is one **big** difference!

The differential vector $\hat{a}_n \, d\ell$ is a vector that tangential to **surface** S (i.e., it lies on surface S), but is **normal** to contour C!

This of course is the **opposite** of the differential vector $\overline{d\ell}$ in that $\overline{d\ell}$ lies **tangential** to the contour:




As a result, we find that $\overline{d\ell} \cdot \hat{a}_n \, d\ell = 0$. However, note the **magnitude** of each vector is identical:

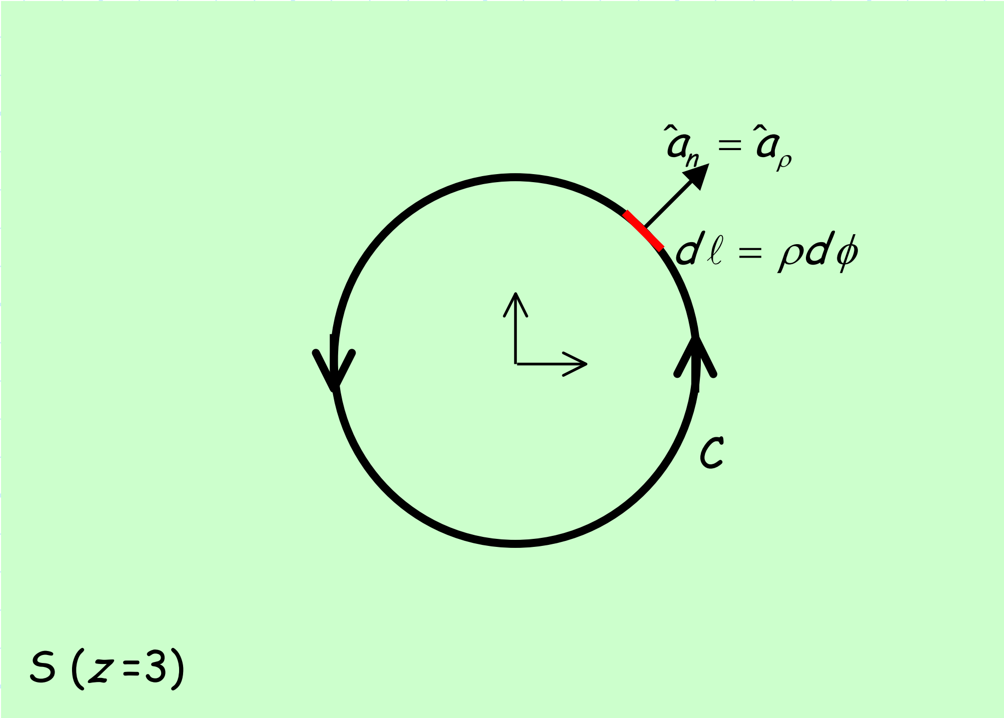
$$\left|\overline{d\ell}\right| = \left|\hat{a}_n \, d\ell\right| = d\ell$$

**For example**, consider the planar surface $z$=3. On this surface is a contour that is a **circle**, radius 2, centered around the $z$-axis.

For the contour integrals we studied in Section 2-5, we would use:

$$\overline{d\ell} = \hat{a}_\phi \, \rho \, d\phi$$

**However**, to determine the total current flowing across the contour, we use $\hat{a}_n = \hat{a}_\rho$ and $d\ell = \rho \, d\phi$. Note the **directions** of these two differential vectors are **different**, but their **magnitudes** are the **same**.



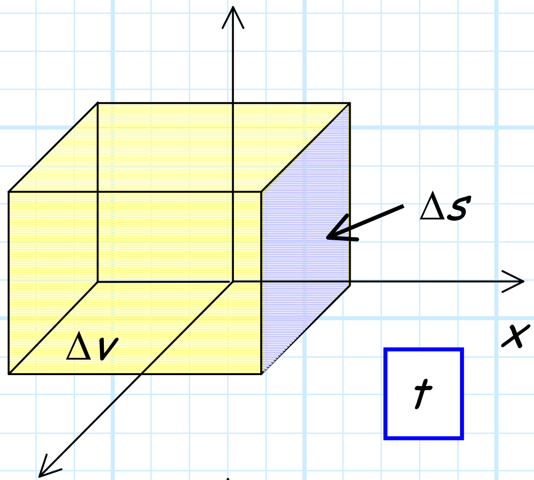

The integral for determining the **total** current flowing from **inside** the circle to **outside** the circle is therefore:

$$\begin{aligned} I &= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_n \, d\ell \\ &= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho \, \rho \, d\phi \\ &= \int_0^{2\pi} \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho 2 \, d\phi \end{aligned}$$

# Charge Velocity and Current Density

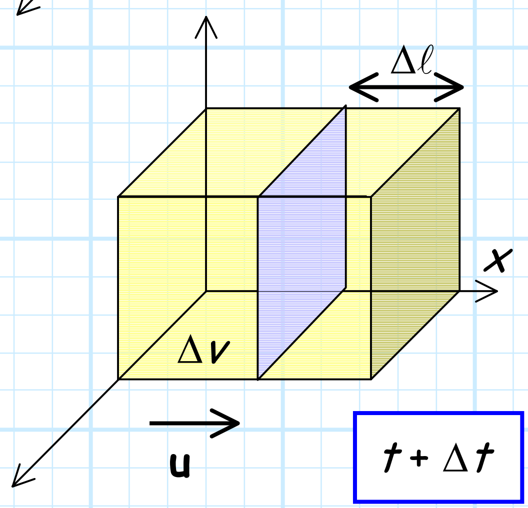
Consider a **small** volume ($\Delta v$) filled with charge Q.

If the charge is **uniformly** distributed, then the **charge density** is:




$$\rho_v(\bar{\mathbf{r}}) = \frac{Q}{\Delta v}$$

Say these charges are **moving** at velocity $\mathbf{u} = u_x\,\hat{a}_x$. Then, in a small **time** $\Delta t$, the charged particles will have moved in the $x$-direction a **distance** $\Delta \ell$:




$$\Delta \ell = u_x\,\Delta t$$

**Q:** *How much **charge** $\Delta Q$ moves across **surface** $\Delta s$ in **time** $\Delta t$ ?*

**A:** The amount is **equal** to the charge occupying **volume** $\Delta s\,\Delta \ell$:

$$\Delta Q = \rho_v(\bar{r})(\Delta s\, \Delta \ell)$$

But remember, $\Delta \ell = u_x\, \Delta t$. Therefore:

$$\Delta Q = \rho_v(\bar{r})\, u_x\, \Delta s\, \Delta t$$

And dividing by $\Delta t$:

$$\frac{\Delta Q}{\Delta t} = \rho_v(\bar{r})\, u_x\, \Delta s$$

Hey! Charge divided by time is equal to **current** !

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v(\bar{r})\, u_x\, \Delta s$$

The current $\Delta I$ is the current flowing **through** the small surface $\Delta s$. We can therefore determine the **current density** on this surface:

$$J_x = \frac{\Delta I}{\Delta s} = \rho_v(\bar{r})\; u_x$$

In other words, current density is equal to the **product** of the charge density and the charge velocity. In general, we can say:

$$\boxed{\mathbf{J}(\bar{r}) = \rho_v(\bar{r})\,\mathbf{u}(\bar{r})}$$

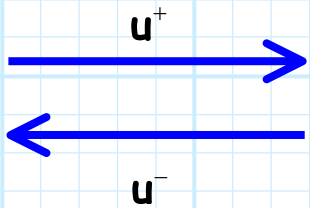
where $\mathbf{u}(\bar{r})$ is a vector field that describes the **velocity** of the moving charge at every point $\bar{r}$.

**IMPORTANT NOTE!** The velocity of charge is **NOT** the speed of light! In fact, charge velocity is generally **nowhere** near $c = 3 \times 10^8$ m/sec (its more like $3 \times 10^{-2}$ m/sec!).

Charge velocity is generally dependent on the **type** of particles that carry the charge (e.g., free electrons, positive ions).

For example, we can denote $\mathbf{u}^+$ the velocity of **positively** charged particles, while $\mathbf{u}^-$ denotes the velocity of **negatively** charged particles.

We find that typically, $\mathbf{u}^+$ and $\mathbf{u}^-$ point in **opposite** directions!




and the velocities will have **unequal** magnitudes:

$$|\mathbf{u}^+| \neq |\mathbf{u}^-|$$

The total current density can therefore be expressed as:

$$\begin{aligned}\mathbf{J}(\bar{r}) &= \mathbf{J}^+(\bar{r}) + \mathbf{J}^-(\bar{r}) \\ &= \rho_v^+(\bar{r})\,\mathbf{u}^+(\bar{r}) + \rho_v^-(\bar{r})\,\mathbf{u}^-(\bar{r})\end{aligned}$$

**Q:** *So, $\mathbf{J}^+(\bar{r})$ and $\mathbf{J}^-(\bar{r})$ must point in opposite directions, since $\mathbf{u}^+(\bar{r})$ and $\mathbf{u}^-(\bar{r})$ point in opposite directions ?*

**A:** NO! It is true that the charges flow in opposite **directions**, but the charges also have opposite **signs** ! Recall $\rho_v^+(\bar{r}) > 0$ and $\rho_v^-(\bar{r}) < 0$, therefore, vectors $\mathbf{J}^+(\bar{r}) = \rho_v^+(\bar{r})\mathbf{u}^+(\bar{r})$ and $\mathbf{J}^-(\bar{r}) = \rho_v^-(\bar{r})\mathbf{u}^-(\bar{r})$ each typically point in the **same** direction!

Remember that, for example, if positive charge is moving **left** and negative charge is moving **right**, then **both** result in **current** flowing toward the **left**.