3-4 The Law of Charge Conservation

Reading Assignment: *pp.68-71*

Q: Charge **conservation**! -

A:

HO: Kirchoff's Current Law

HO: The Continuity Equation

HO: The Point Form Continuity Equation

Kirchoff's Current Law

So, we now know that:

$$I = \frac{dQ}{dt} = \iint_S \mathbf{J}(\bar{r}) \cdot \overline{ds}$$

Consider now the case where S is a **closed** surface:

$$I = \frac{dQ}{dt} = \oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds}$$

The current I thus describes the rate at which **net** charge is **leaving** some **volume V** that is surrounded by surface S.

We will find that **often** this rate is I=**0** !

Q: *Yikes! Why would this value be zero??*

A: Because charge can be neither **created** nor **destroyed**!

Think about it.

If there was some **endless** flow of charge crossing closed surface S—**exiting** volume V—then there would have to be some "fountain" of charge creating this endless **outward** flow.

Alternatively, if there was some **endless** flow of charge crossing closed surface S—**entering** volume V—then there would have to be some charge "drain" that disposed of this endless **inward** flow.

* But, we **cannot** create or destroy charge—**endless** charge fountains or charge drains **cannot** exist!

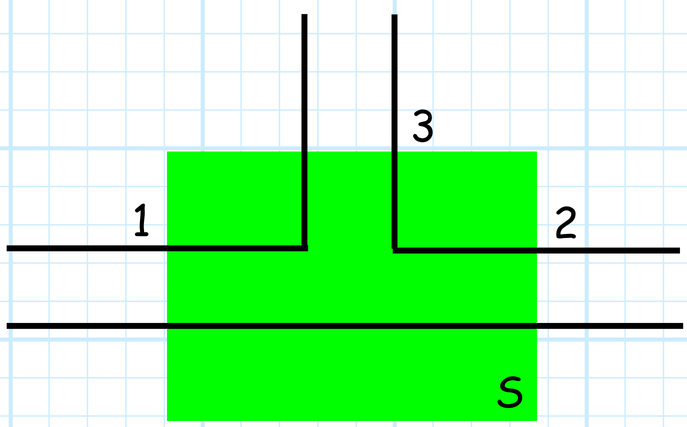
* Instead, charge **exiting** volume V through surface S must have likewise **entered** volume V through surface S (and vice versa).

* As a result, the rate of net charge flow (i.e., **current**) across a **closed** surface is very often **zero**!

In other "words", we can state:

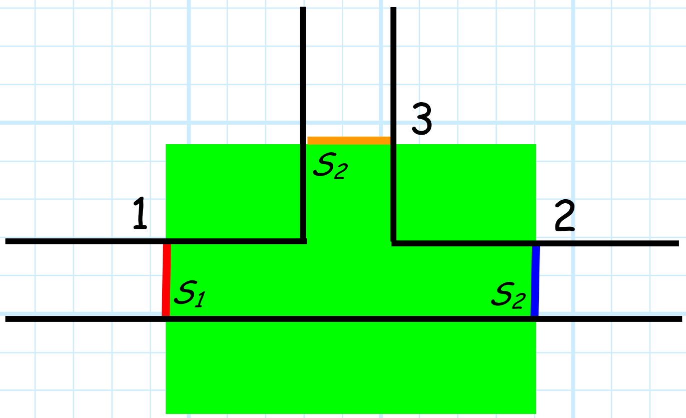
$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$$

For example, consider a closed surface S that surrounds a "node" at which 3 conducting **wires** converge:



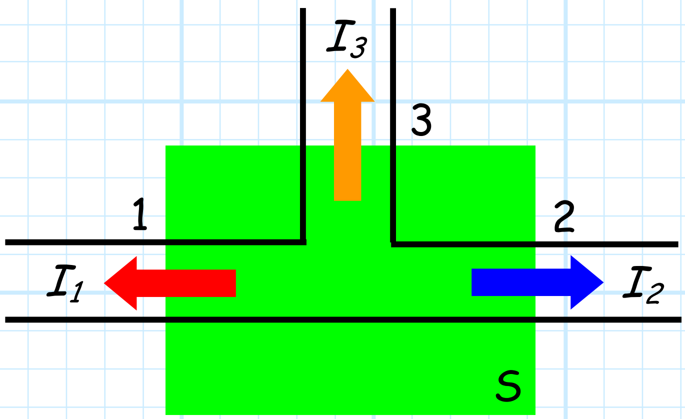
Since current is flowing **only** in these wires, the surface integral reduces to a surface integration over the cross section of **each** of the three wires:

$$\oiint_S \mathbf{J}(\bar{r}_s) \cdot \overline{ds} = \oiint_{S_1} \mathbf{J}(\bar{r}_s) \cdot \overline{ds_1} + \oiint_{S_2} \mathbf{J}(\bar{r}_s) \cdot \overline{ds_2} + \oiint_{S_3} \mathbf{J}(\bar{r}_s) \cdot \overline{ds_3}$$



The result of each integration is simply the **current** flowing in each wire!

$$\oiint_S \mathbf{J}(\bar{r}_s) \cdot \overline{ds} = I_1 + I_2 + I_3$$



But remember, since we know that charge cannot be created or destroyed, we have concluded that:

$$\oiint_S \mathbf{J}(\bar{r}_s) \cdot \overline{ds} = 0$$

Meaning:

$$0 = I_1 + I_2 + I_3$$

More generally, if this node had n wires, we could state that:

$$\boxed{0 = \sum_n I_n}$$

Hopefully you recognize this statement—it's **Kirchoff's Current Law**!

Therefore, a more general, **electromagnetic** expression of Kirchoff's Current Law is:

$$\boxed{\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = 0}$$

Note that this result means that the current density $\mathbf{J}(\bar{r})$ (for this case) is **solenoidal**!

In other words, the above integral **likewise** means that $\nabla \cdot \mathbf{J}(\bar{r}) = 0$.

Gustav Robert Kirchhoff (1824-1887), German physicist, announced the laws that allow calculation of the currents, voltages, and resistances of electrical networks in 1845, when he was only **twenty-one** (so what have **you** been doing)! His other work established the technique of spectrum analysis that he applied to determine the composition of the Sun.

The Continuity Equation

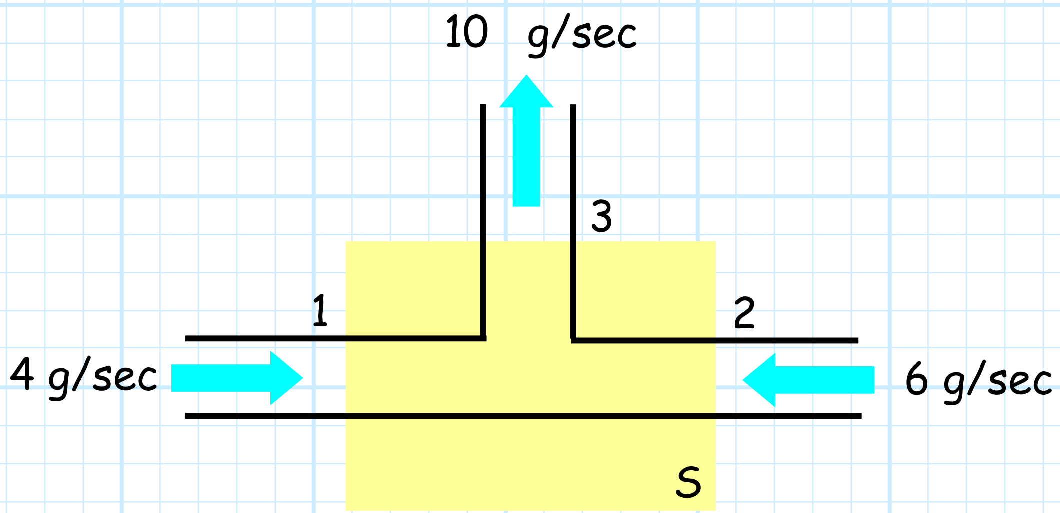
For **some** closed surfaces,

$$I = \oint\!\!\oint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = \frac{dQ}{dt} \neq 0 \quad !!!$$

Q: *How is this possible ? I thought you said charge cannot be* ***created*** *or* ***destroyed****.*

A: Let's try this **analogy**.

Say we have three pipes that carry **water** to/from a node:



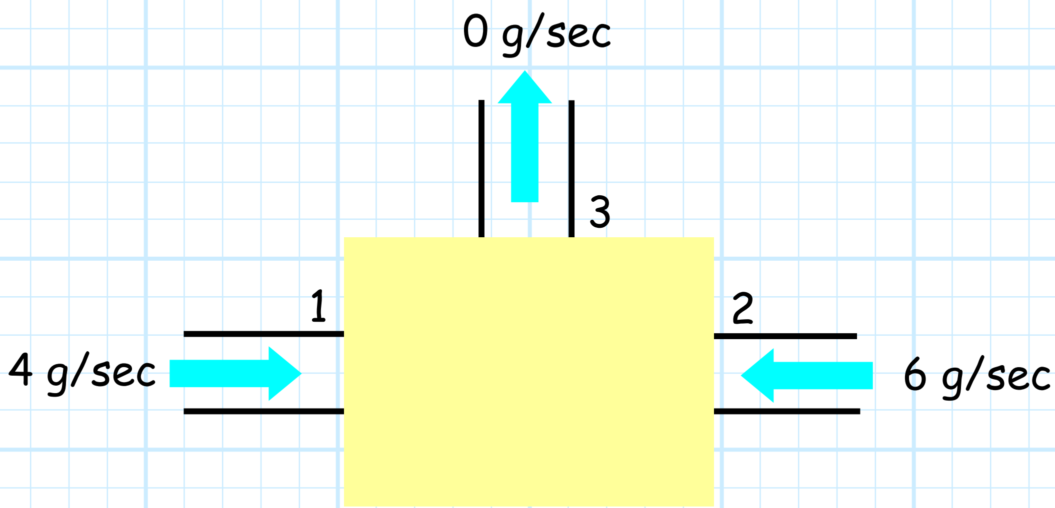
If a current of **4** gallons/second **enters** the node through pipe 1, and another **6** gallons/second **enters** through pipe 2, then **10** gallons/second **must** be **leaving** the node through pipe 3.

The reason for this of course is that water cannot be **created** or **destroyed**, and therefore if water **enters** surface S at a rate of 10 gallons/sec, then water must also **leave** at the same rate.

Therefore, the **amount of water** $W(t)$ in closed surface S remains **constant** with time. I.E.,

$$\frac{d\,W(t)}{dt} = 0$$

Now, consider the system below. Water is entering through pipe 1 and pipe 2, **again** at a rate of 4 gallons/second and 6 gallons/second, respectively.

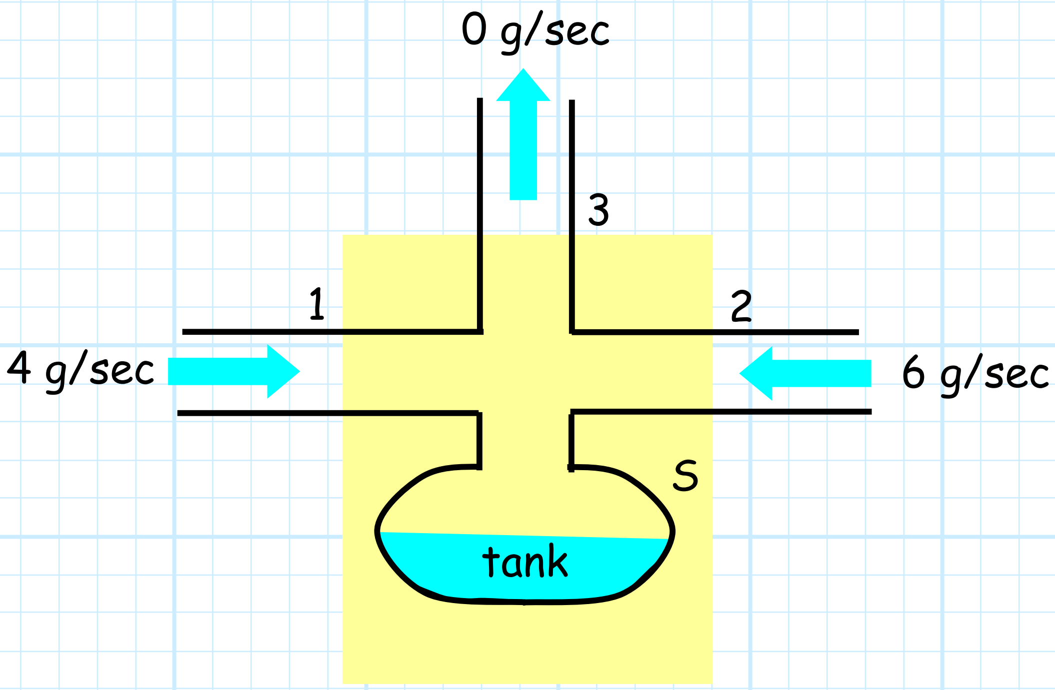


However, this time we find that **no** water is leaving through pipe 3! Therefore:

$$\frac{d\,W(t)}{dt} \neq 0$$

Q: *How is this possible? What happens to the water?*

A: It's possible because the closed surface S surrounds a **storage device** (i.e., a water **tank**).



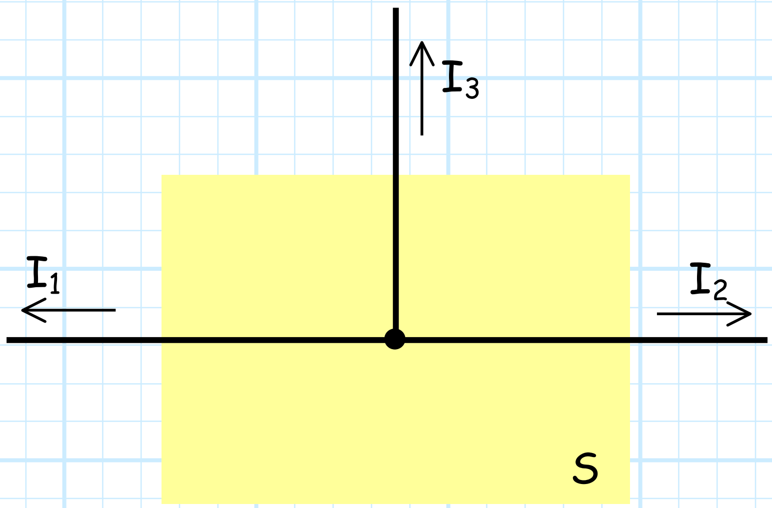
In addition to being a **sink** for water, this tank can also be a **source**. As a result, the current exiting pipe 3 could also **exceed** 10 gallons/second !

The "catch" here is that this **cannot last forever**. Eventually, the tank will get completely **full** or completely **empty**. After that we will find again that $dW(t)/dt = 0$.

Now, let's return to **charge**.

It would likewise **appear** that the charge **enclosed** ($Q_{enc}(t)$) within some surface that surrounds a circuit node must **always** be constant with respect to time. I.E.,

$$\frac{d\,Q_{enc}(t)}{dt} = 0$$



Therefore:

$$I = \oint\!\!\oint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = 0$$

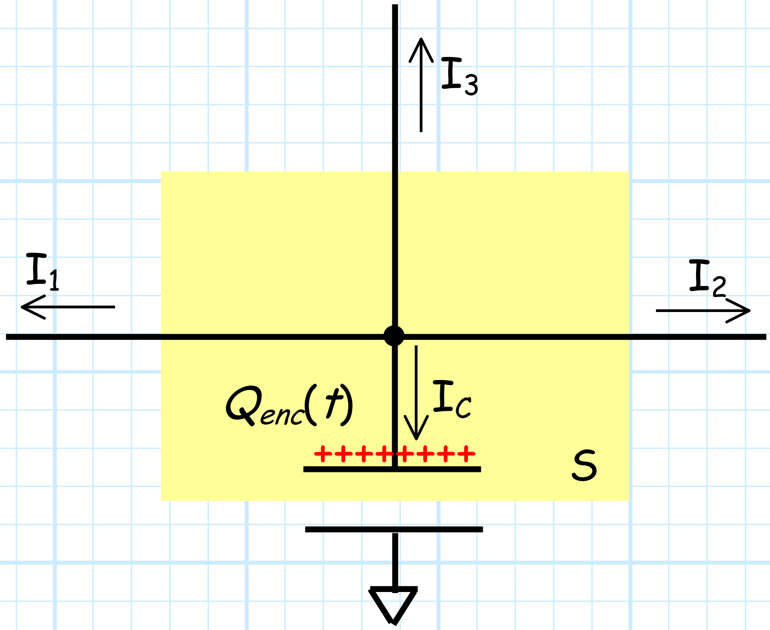
or

$$\sum_{n=1}^{N} I_n = 0$$

But, there is such a thing as a charge "tank"!

A charge tank is a **capacitor**.

A capacitor can either store or source **enclosed** charge $Q_{enc}(t)$, such that $d\,Q_{enc}(t)/dt \neq 0$.



The current I_C is known as **displacement current.** We find that:

$$I_C = -\sum_{n=1}^{N} I_n$$

Meaning of course that KCL must **also** include displacement current:

$$\boxed{I_C + \sum_{n=1}^{N} I_n = 0}$$

Now, recall again that:

$$\oiint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = \frac{d\,Q(t)}{dt}$$

where *Q(t)* represents the charge moving **from** the inside of surface *S* **to** the outside the surface *S*.

Note an increase in the charge **outside** the surface S results in a corresponding decrease in the total charge **enclosed** by S (i.e., $Q_{enc}(t)$). Therefore:

$$\frac{d\,Q(t)_{enc}}{dt} = -\frac{d\,Q(t)}{dt}$$

If these derivatives are not zero, then **displacement current** must exist with in volume surrounded by S!

The **value** of this displacement current is equal to $dQ_{enc}(t)/dt$.

Thus, if **displacement** current exists (meaning that there is some way to "store" charge) the **continuity equation** becomes:

$$\boxed{\oint\!\!\oint_S \mathbf{J}(\bar{\mathrm{r}})\cdot\overline{ds} = -\frac{dQ_{enc}(t)}{dt} = -I_C}$$

Note this means that the current flowing **out** of surface S (i.e., I) is equal to the **opposite** value of displacement current $dQ_{enc}(t)/dt$.

This of course means that the current **entering** surface S (i.e., -I) is **equal** to the displacement current $dQ_{enc}(t)/dt$.

Makes sense! If the total current flowing **into** a closed surface S is **positive**, then the total charge enclosed by the surface is **increasing.** This charge must all be stored somewhere, as it cannot be destroyed!

The continuity equation can therefore alternatively be written as:

$$\boxed{\begin{gathered}\oint\!\!\oint_S \mathbf{J}(\bar{\mathrm{r}})\cdot \overline{ds} + \frac{dQ_{enc}(t)}{dt} = 0 \\[2em] \oint\!\!\oint_S \mathbf{J}(\bar{\mathrm{r}})\cdot \overline{ds} + I_C = 0\end{gathered}}$$

If displacement current does **not** exist, then $d\,Q_{enc}(t)/dt = 0$ and the continuity equation remains:

$$I = \oint\!\!\oint_S \mathbf{J}(\bar{\mathrm{r}})\cdot \overline{ds} = 0$$

The Point Form Continuity Equation

Recall that the charge **enclosed** in a volume V can be determined from the **volume charge density**:

$$Q_{enc} = \iiint_V \rho_v(\bar{\mathrm{r}})\, dv$$

If charge is **moving** (i.e., current flow), then charge density **can** be a function of **time** (i.e., $\rho_v(\bar{\mathrm{r}},\mathrm{t})$). As a result, we write:

$$Q_{enc}(t) = \iiint_V \rho_v(\bar{\mathrm{r}},t)\, dv$$

Inserting this into the **continuity equation**, we get:

$$\begin{aligned} \oiint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds} &= -\frac{d\, Q_{enc}(t)}{dt} \\ &= -\frac{d}{dt} \iiint_V \rho_v(\bar{\mathrm{r}},t)\, dv \end{aligned}$$

where closed surface S **surrounds** volume V.

Now recall the **divergence theorem**! Using this theorem, know that:

$$\oiint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds} = \iiint_V \nabla \cdot \mathbf{J}(\bar{\mathrm{r}})\, dv$$

Combining this with the continuity equation, we find:

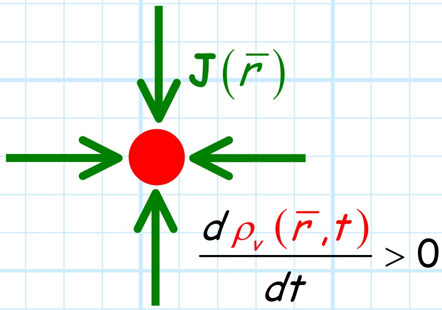
$$\iiint_V \nabla \cdot \mathbf{J}(\bar{\mathrm{r}})\, dv = -\frac{d}{dt}\iiint_V \rho_v(\bar{\mathrm{r}},t)\, dv$$

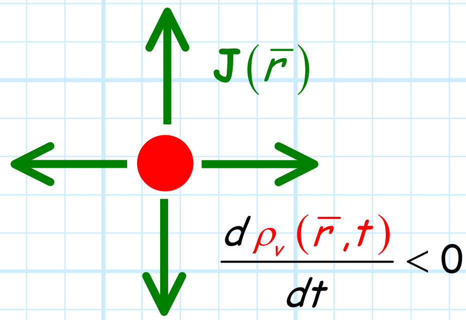
From this equation, we can conclude:

$$\boxed{\nabla \cdot \mathbf{J}(\bar{\mathrm{r}}) = -\frac{d\,\rho_v(\bar{\mathrm{r}},t)}{dt}}$$

This is the **point form** of the continuity equation. It says that if the **density** of charge at some point $\bar{\mathrm{r}}$ is **increasing** with time, then **current** must be **converging** to that point.

Or, if charge density is **decreasing** with time, then current is **diverging** from point $\bar{r}$.





Current is **converging** on point, therefore charge density is **increasing**.

Current is **diverging** from point, therefore charge density is **decreasing**.

Notice that the scalar field:

$$\frac{d\rho_v(\bar{r},t)}{dt}$$

describes the **rate** at which the charge density is increasing at each and every point in the universe!

For **example**, say the divergence of $\mathbf{J}(\bar{r})$ Amps/m^2, when evaluated at some point denoted by position vector $\bar{r}_a$, is equal to 3.0:

$$-\nabla\cdot\mathbf{J}(\bar{r})\Big|_{\bar{r}=\bar{r}_a} = 3 = \frac{d\rho(\bar{r}_a,t)}{dt} \quad \frac{Amps}{m^3}$$

This means that the charge density at point $\bar{r}_a$ is increasing at a **rate** of 3 coulombs/m^3 every second!

E.G.: In **4** seconds, the charge density at $\bar{r}_a$ will **increase** by a value of **12** C/m^3.

Note the equation:

$$-\nabla \cdot \mathbf{J}(\bar{r})\Big|_{\bar{r}=\bar{r}_a} = 3 = \frac{d\,\rho(\bar{r}_a,t)}{dt}$$

is a **differential equation**. Our task is to **find** the function $\rho(\bar{r}_a,t)$, given that we know its time derivative is equal to 3.0.

The solution for this **example** can be found by **integrating** both sides of the equation (with respect to time), i.e.:

$$\rho(\bar{r}_a,t) = 3t + \rho(\bar{r}_a,t=0) \qquad \frac{C}{m^3}$$

where $\rho(\bar{r}_a,t=0)$ simply indicates the value of the charge density at point $\bar{r}_a$, at time t = 0. The value 3t is then the additional charge density (beyond $\rho(\bar{r}_a,t=0)$) at time t.