4-7 Poisson's and Laplace's Equations

Reading Assignment: pp. 116-

\[ E(\vec{r}) = -\nabla V(\vec{r}) \]

\[ \nabla \times E(\vec{r}) = 0 \quad \nabla \cdot E(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \]

Q:

A:

HO: Poisson's and Laplace's Equations
Poisson's and Laplace's Equation

We know that for the case of static fields, Maxwell's Equations reduces to the electrostatic equations:

\[
\nabla \times \mathbf{E}(\vec{r}) = 0 \quad \text{and} \quad \nabla \cdot \mathbf{E}(\vec{r}) = \frac{\rho_v(\vec{r})}{\varepsilon_0}
\]

We can alternatively write these equations in terms of the electric potential field \( V(\vec{r}) \), using the relationship \( \mathbf{E}(\vec{r}) = -\nabla V(\vec{r}) \):

\[
-\nabla \times \nabla V(\vec{r}) = 0 \quad \text{and} \quad -\nabla \cdot \nabla V(\vec{r}) = \frac{\rho_v(\vec{r})}{\varepsilon_0}
\]

Let's examine the first of these equations. Recall that we determined in Chapter 2 that:

\[
\nabla \times \nabla g(\vec{r}) = 0
\]

for any and all scalar functions \( g(\vec{r}) \). In other words the first equation is simply a mathematical identity—it says nothing physically about the electric potential field \( V(\vec{r}) \)!
The second equation includes the operation $\nabla \cdot \nabla$. We recall from Chapter 2 that this operation is called the scalar Laplacian:

$$\nabla \cdot \nabla = \nabla^2$$

Therefore, we can write the relationship between charge density and the electric potential field in terms of one equation!

$$\nabla^2 V(\vec{r}) = -\frac{\rho_v(\vec{r})}{\varepsilon_0}$$

This equation is known as Poisson’s Equation, and is essentially the “Maxwell’s Equation” of the electric potential field $V(\vec{r})$.

Note that for points where no charge exist, Poisson’s equation becomes:

$$\nabla^2 V(\vec{r}) = 0$$

This equation is know as Laplace’s Equation. Although it looks very simple, most scalar functions will not satisfy Laplace’s Equation! Only a special class of scalar fields, called analytic functions will satisfy Laplace’s equation.