5-2 Conductors

Reading Assignment: *pp. 122-132*

We have been studying the electrostatics of **free-space** (i.e., a vacuum).

But, the universe is full of **stuff!**

Q: Does stuff (material) affect our electrostatics knowledge?

A:

HO: Dielectrics and Conductors

A. Ohm's Law

So, in conductors, applying an electric field $\mathbf{E}(\bar{r})$ will cause current $\mathbf{J}(\bar{r})$ to form.

Q:

A: HO: Ohm's Law

B. Resistance

Q: I thought Ohm's Law was *R=V/I*?

A:

HO: Resistors

C. Perfect Conductors

Consider now a perfect conductor (i.e., $\sigma = \infty$).

Q: Does this mean that current density $\mathbf{J}(\bar{r})$ is likewise infinite?

A:

HO: Perfect Conductors

D. Kirchoff's Voltage Law

Recall since a static field $\mathbf{E}(\bar{r})$ is conservative:

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

$$\sum_{v} V_n = 0$$

HO: Kirchoff's Voltage Law

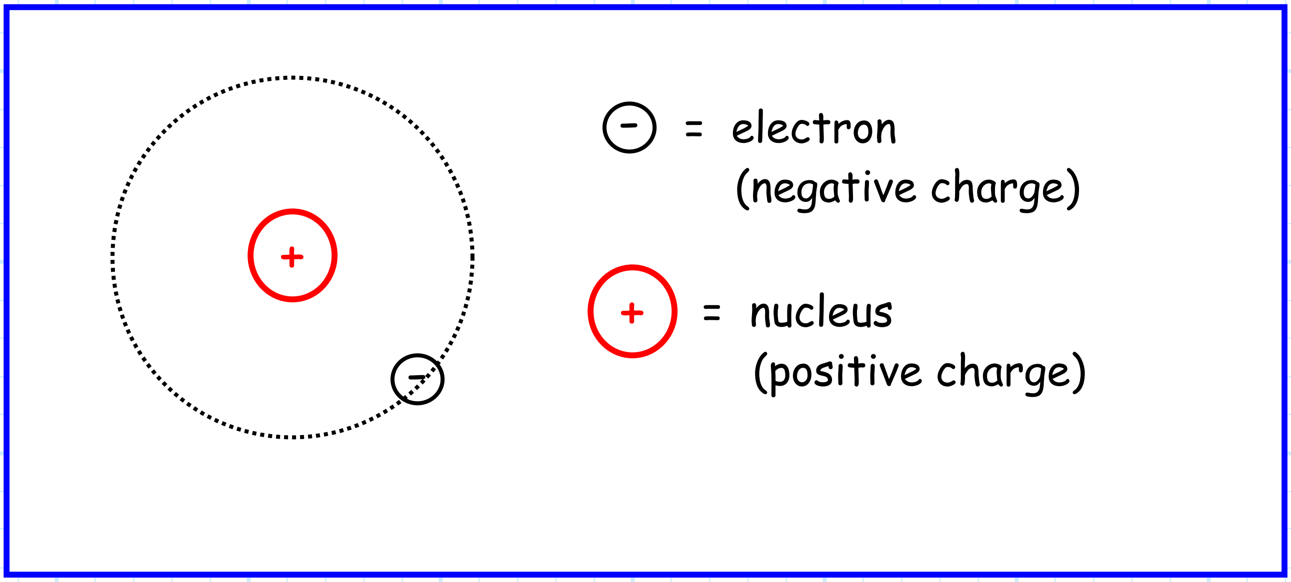
E. Joule's Law

Conducting material will **absorb** energy—it will **heat** up!

HO: Joule's Law

Dielectrics and Conductors

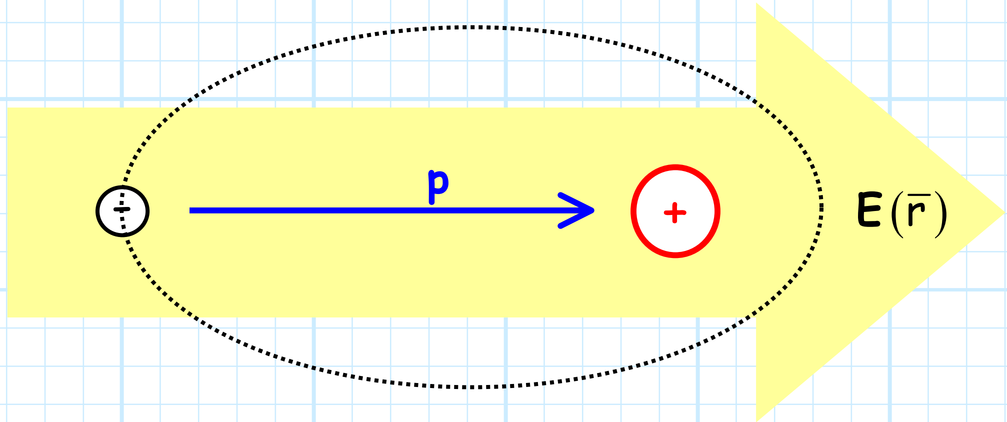
Consider a very **simple** model of an **atom**:



Say an **electric field** is applied to this atom.

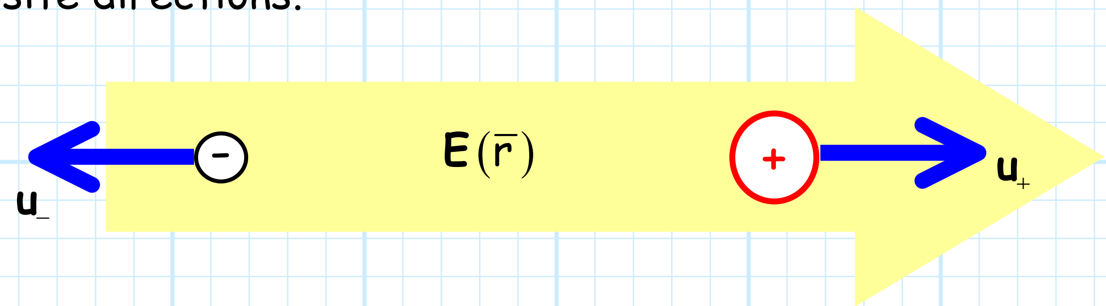
Note the field will apply a **force** on both the positively charged nucleus and the negatively charged electron. However, these forces will move these particles in **opposite** directions!

Two things may occur. In the **first** case, the atom may **stretch**, but the electron will remain **bound** to the atom:



Note, an **electric dipole** has been created !

For the **second** case, the electron may be **break free** from the atom, creating a positive ion and a **free electron**. We call these free charges, and the electric field will cause them to **move** in opposite directions.



Moving charge! We know what moving charge is. Moving charge is **electric current** $\mathbf{J}(\bar{r})$.

These two examples provide a simple demonstration of what occurs when an electric field is applied to some **material** (e.g., plastic, copper, water, oxygen).

1) Materials where the charges remain bound (and thus dipoles are created) are called **insulator** (or **dielectric**) materials.

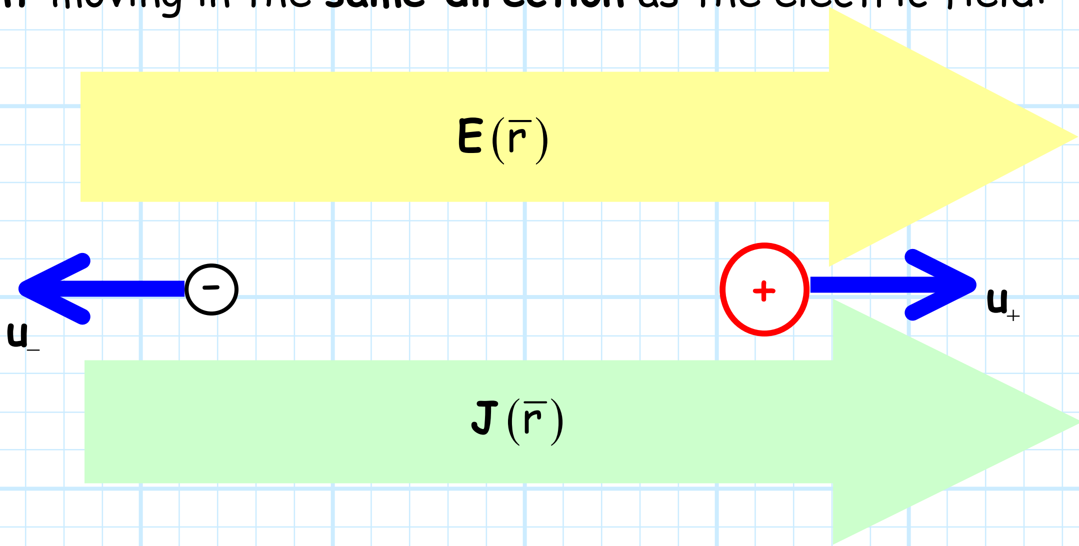
2) Materials where the electrons are free to move are called **conductors**.

Of course, materials consists of molecules with **many electrons**, and in general some electrons are **bound** and some are **free**. As a result, there are no **perfect** conductors or **perfect** insulators, although some materials are **very** close!

Additionally, some materials are lie between being a good conductor or a good insulator. We call these materials **semi-conductors** (e.g., Silicon).

Ohm's Law

Recall that a positively charge particle will move in the direction of an electric field, whereas a negative chare will move in the opposite direction. Both types of charge, however, result in **current** moving in the **same direction** as the electric field:



Q: *So, the direction of current density $\mathbf{J}(\bar{r})$ and electric field $\mathbf{E}(\bar{r})$ are the same. The question then is, how are their **magnitudes** related?*

A: They are related by **Ohm's Law**:

$$\mathbf{J}(\bar{r}) = \sigma(\bar{r})\,\mathbf{E}(\bar{r})$$

The scalar value $\sigma(\bar{r})$ is called the material's **conductivity**. Note the units of conductivity are:

$$\sigma(\bar{r}) = \frac{\mathbf{J}(\bar{r})}{\mathbf{E}(\bar{r})} \qquad \left(\frac{\text{Amps}}{\text{m}^2}\right)\left(\frac{\text{Volts}}{\text{m}}\right)^{-1}$$

$$= \frac{\mathbf{J}(\bar{r})}{\mathbf{E}(\bar{r})} \qquad \left(\frac{\text{Amps}}{\text{m}^2}\right)\left(\frac{\text{m}}{\text{Volts}}\right)$$

$$= \frac{\mathbf{J}(\bar{r})}{\mathbf{E}(\bar{r})} \qquad \left(\frac{\text{Amps}}{\text{Volts}\cdot\text{m}}\right)$$

$$= \frac{\mathbf{J}(\bar{r})}{\mathbf{E}(\bar{r})} \qquad \left(\frac{1}{\text{Ohm}\cdot\text{m}}\right)$$

In other words, the unit of conductivity is **conductance/unit length.**

We emphasize that conductivity $\sigma(\bar{r})$ is **a material parameter**. For example, the conductivity of **copper** is:

$$\sigma_{copper} = 5.8 \times 10^{7} \left[\frac{1}{\Omega\,\text{m}}\right]$$

and the conductivity of **polyethylene** (a plastic) is:

$$\sigma_{polyethylene} = 1.5 \times 10^{-12} \left[\frac{1}{\Omega\,\text{m}}\right]$$

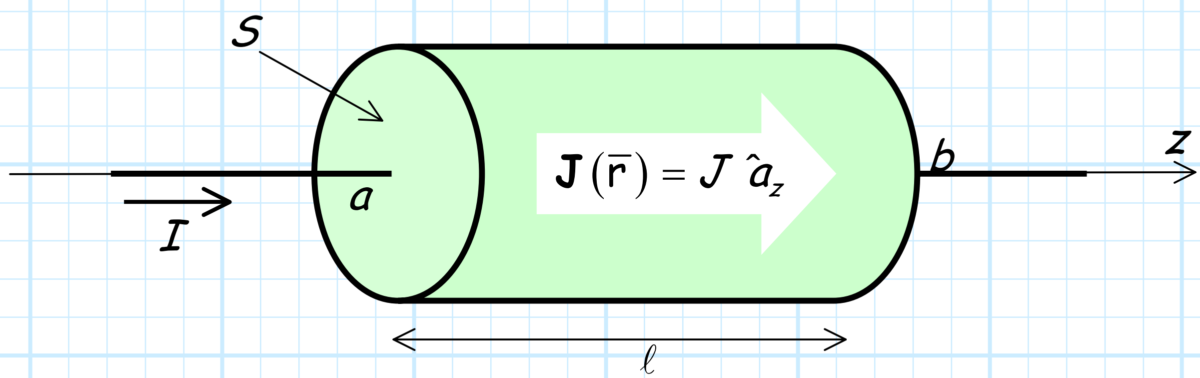
Note the **vast** difference in conductivity between these two materials. Copper is a **conductor** and polyethylene is an **insulator**.

Georg Simon Ohm (1789-1854) was the German physicist who in 1827 discovered the law that the current flow through a conductor is proportional to the voltage and inversely proportional to the resistance. Ohm was then a professor of mathematics in Cologne. His work was **coldly** received! The Prussian minister of education announced that "a professor who preached such heresies was unworthy to teach science." Ohm resigned his post, went into academic exile for several years, and then left Prussia and became a professor in Bavaria.

From: www.ee.umd.edu/~taylor/frame2.htm

Resistors

Consider a **uniform** cylinder of material with mediocre to poor to pathetic **conductivity** $\sigma(\bar{r}) = \sigma$.



This cylinder is centered on the *z*-axis, and has **length** ℓ. The **surface area** of the ends of the cylinder is *S*.

Say the cylinder has **current** *I* flowing into it (and thus out of it), producing a current **density** $\mathbf{J}(\bar{r})$.

By the way, this cylinder is commonly referred to as a **resistor**!

Q: *What is its* ***resistance*** *R of this resistor, given length* ℓ*, cross-section area S, and conductivity* σ*?*

A: Let's first begin with the circuit form of Ohm's Law:

$$R = \frac{V}{I}$$

where V is the potential difference between the two ends of the resistor (i.e., the voltage across the resistor), and I is the current through the resistor.

From **electromagnetics**, we know that the potential difference V is:

$$V = V_{ab} = \int_a^b \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell}$$

and the current I is:

$$I = \iint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds}$$

Thus, we can **combine** these expressions and find resistance R, expressed in terms of electric field $\mathbf{E}(\bar{\mathrm{r}})$ within the resistor, and the current density $\mathbf{J}(\bar{\mathrm{r}})$ within the resistor:

$$R = \frac{V}{I} = \frac{\int_a^b \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell}}{\iint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds}}$$

Lets evaluate **each integral** in this expression to determine the resistance R of the device described earlier!

1) The voltage V is the potential difference V_{ab} between point a and point b:

$$V = V_{ab} = \int_a^b \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell}$$

Q: *But, what is the electric field* $\mathbf{E}(\bar{\mathbf{r}})$*?*

A: The electric field within the resistor can be determined from **Ohm's Law**:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{\mathbf{J}(\bar{\mathbf{r}})}{\sigma(\bar{\mathbf{r}})}$$

We can assume that the **current density** is approximately **constant** across the cross section of the cylinder:

$$\mathbf{J}(\bar{\mathbf{r}}) = J\, \hat{a}_z$$

Likewise, we know that the conductivity of the resistor material is a **constant**:

$$\sigma(\bar{\mathbf{r}}) = \sigma$$

As a result, the electric field **within** the resistor is:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{\mathbf{J}(\bar{\mathbf{r}})}{\sigma(\bar{\mathbf{r}})} = \frac{J}{\sigma}\, \hat{a}_z$$

Therefore, **integrating** in a straight line along the *z*-axis from point *a* to point *b*, we find the potential difference *V* to be:

$$\begin{aligned} V &= \int_a^b \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= \frac{1}{\sigma} \int_{z_a}^{z_b} J \ \hat{a}_z \cdot \hat{a}_z \, dz \\ &= \frac{J}{\sigma} \int_{z_a}^{z_b} dz \\ &= \frac{J\ell}{\sigma} \end{aligned}$$

2) We likewise know that the current *I* through the resistor is found by evaluating the **surface integral**:

$$\begin{aligned} I &= \iint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds} \\ &= \iint_S J \ \hat{a}_z \cdot \hat{a}_z \, ds_z \\ &= J \iint_S ds_z \\ &= J \ S \end{aligned}$$

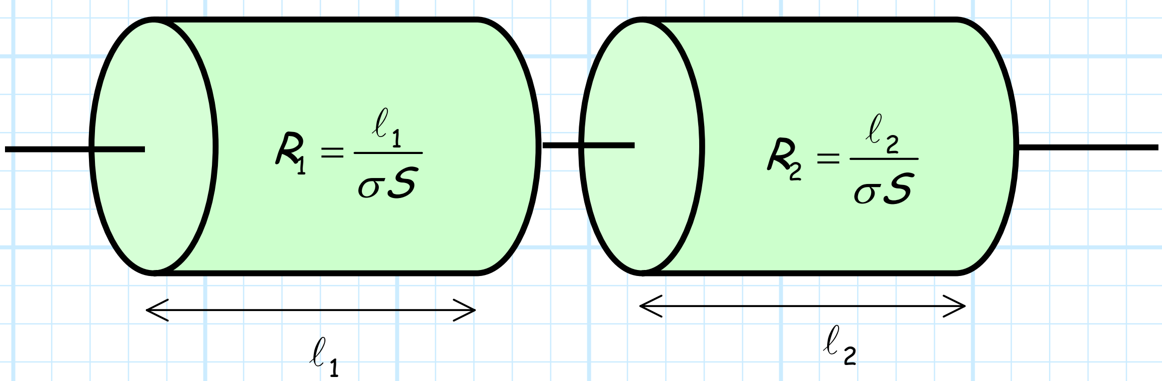
Therefore, the resistance *R* of **this particular** resistor is:

$$
\begin{aligned}
R &= \frac{V}{I} \\
&= \left(\frac{J\ell}{\sigma}\right)\left(\frac{1}{J\,S}\right) \\
&= \frac{\ell}{\sigma S}
\end{aligned}
$$

An interesting result! Consider a resistor as sort of a "clogged pipe". **Increasing** the cross-sectional area S makes the pipe bigger, allowing for **more current** flow. In other words, the resistance of the pipe decreases, as predicted by the above equation.

Likewise, increasing the **length** ℓ simply increases the length of the "clog". The current encounters resistance for a longer distance, thus the value of R increases with increasing length ℓ. Again, this behavior is predicted by the equation shown above.

For **example**, consider the case where we add two resistors together:

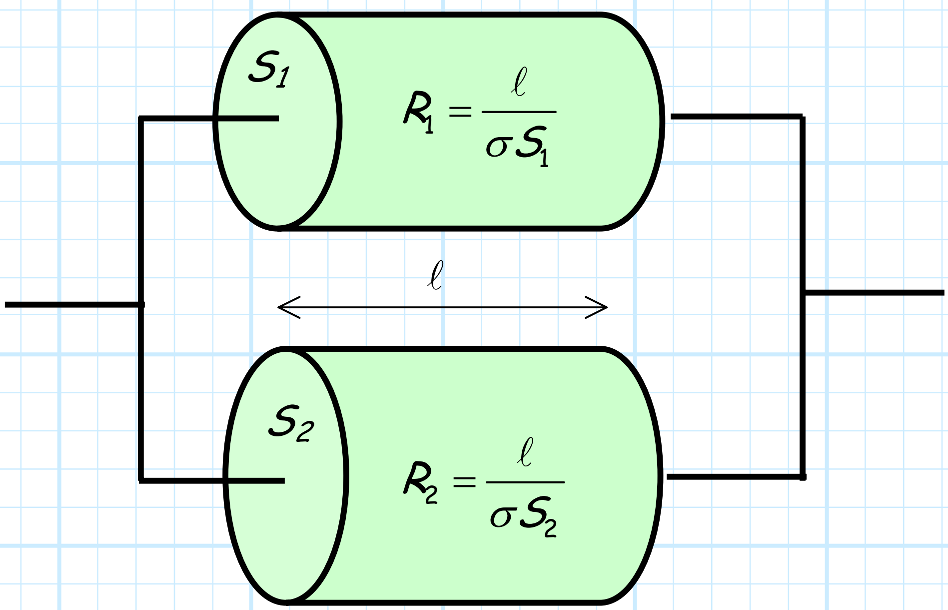


We can view this case as a single resistor with a length $\ell_1 + \ell_2$, resulting in a total resistance of:

$$
\begin{aligned}
R_{total} &= \frac{\ell_1 + \ell_2}{\sigma S} \\
&= \frac{\ell_1}{\sigma S} + \frac{\ell_2}{\sigma S} \\
&= R_1 + R_2
\end{aligned}
$$

But, this result is not the **least bit** surprising, as the two resistors are connected in **series**!

Now let's consider the case where two resistors are connected in a different manner:



We can view this as a single resistor with a total cross sectional area of $S_1 + S_2$. Thus, its total resistance is:

$$
\begin{aligned}
R_{total} &= \frac{\ell}{\sigma\left(S_1 + S_2\right)} \\
&= \left[\frac{\sigma\left(S_1 + S_2\right)}{\ell}\right]^{-1} \\
&= \left[\frac{\sigma S_1}{\ell} + \frac{\sigma S_2}{\ell}\right]^{-1} \\
&= \left[\frac{1}{R_1} + \frac{1}{R_2}\right]^{-1}
\end{aligned}
$$

Again, this should be no surprise, as these two resistors are connected in **parallel**.

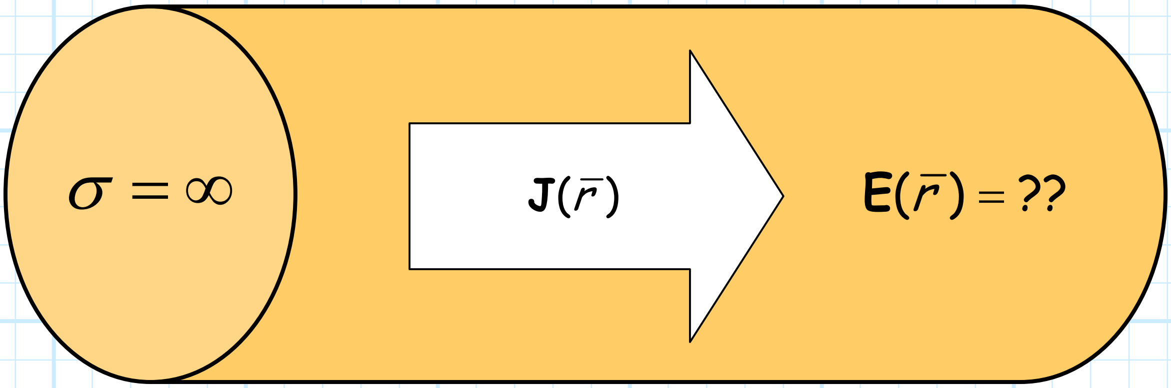
IMPORTANT NOTE: The result $R = \ell / \sigma S$ is valid **only** for the resistor described in this handout. Most importantly, it is valid only for a resistor whose conductivity is a **constant** ($\sigma(\bar{r}) = \sigma$).

If the conductivity is **not** a constant, then we **must** evaluate the potential difference across the resistor with the more **general** expression:

$$
\begin{aligned}
V_{ab} &= \int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} \\
&= \int_a^b \frac{\mathbf{J}(\bar{r})}{\sigma(\bar{r})} \cdot \overline{d\ell}
\end{aligned}
$$

Perfect Conductors

Consider now some current with density $\mathbf{J}(\bar{r})$, flowing within some material with **perfect conductivity** (i.e., $\sigma = \infty$)!



Q: *What is the* ***electric field*** $\mathbf{E}(\bar{r})$ *within this perfectly conducting material?*

A: Well, we know from **Ohm's Law** that the electric field is to the material conductivity and current density as:

$$\mathbf{E}(\bar{r}) = \frac{\mathbf{J}(\bar{r})}{\sigma}$$

Thus, as the material conductivity approaches **infinity**, we find:

$$\lim_{\sigma \to \infty} \mathbf{E}(\bar{r}) = \frac{\mathbf{J}(\bar{r})}{\sigma} = 0$$

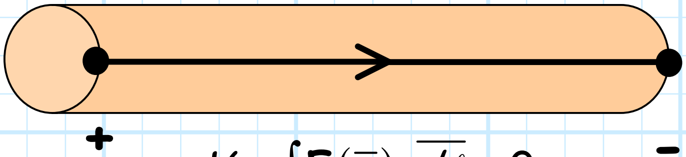
The **electric field** in within a perfectly conducting material is always equal to **zero**!

This makes since when you think about it! Since the material offers **no resistance**, we can move charges through it **without** having to apply any **force** (i.e., and electric field).

This is just like a skater moving across ***frictionless*** *ice! I can continue to move with great velocity, even though* ***no force*** *is being applied!*

Consider what this means with regards to a **wire** made of a **perfectly conducting** material (an often applied assumption).

The electric potential difference between either end of a perfectly conducting wire is **zero**!

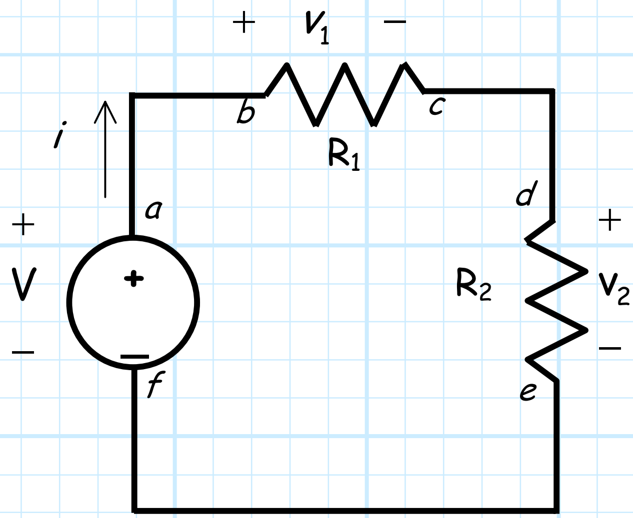


$$V = \int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

Since the electric field within a perfect wire is **zero**, the voltage across any perfect wire is also **zero**, regardless of the current flowing through it.

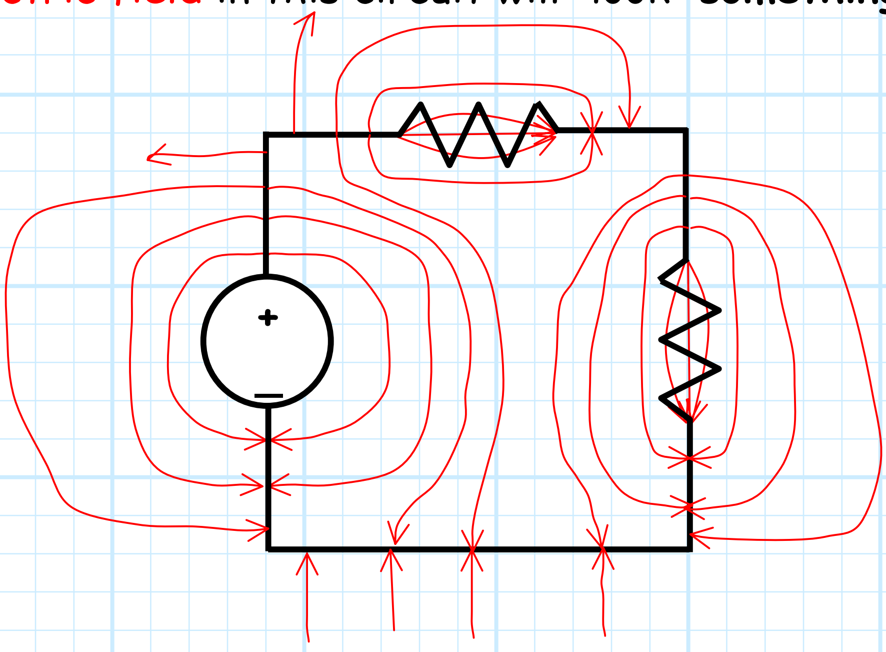
Kirchoff's Voltage Law

Consider a simple electrical **circuit**:



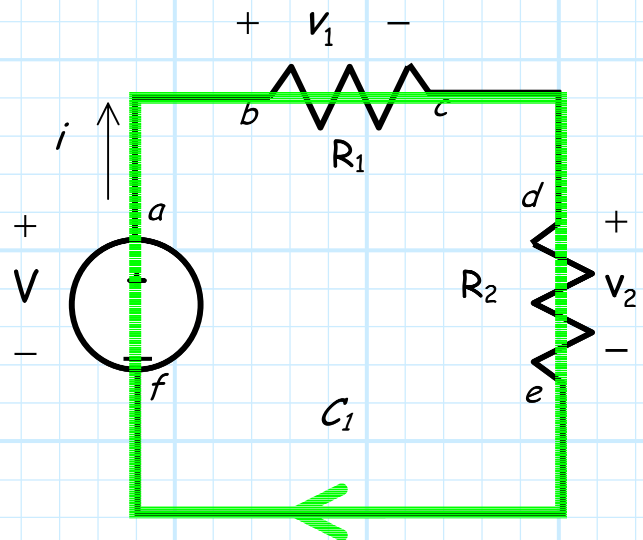
We find that if the **voltage source** is on (i.e., $V \neq 0$), then there will be electric potential differences (i.e., voltage) between different points of the circuit. This can **only** be true if **electric fields** are present!

The electric field in this circuit will "look" **something** like this:



So, instead of using circuit theory, let's use our new **electromagnetic** knowledge to **analyze** this circuit.

First, consider a **contour** C_1 that follows the circuit path.



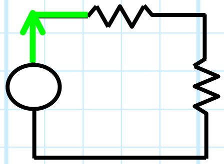
Using this path, let's **evaluate** the contour integral:

$$\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

This is most easily done by breaking the contour C_1 into **six sections**: section 1 extends from point a to point b, section 2 extends from point b to point c, etc. Thus, the integral becomes:

$$\begin{aligned}\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = &\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_b^c \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_c^d \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \\ &\int_d^e \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_e^f \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_f^a \mathbf{E}(\bar{r}) \cdot \overline{d\ell}\end{aligned}$$

Let's evaluate each term **individually**:

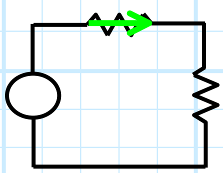


Section 1 (*a* to *b*)

In this section, the contour follows the **wire** from the voltage source to the first resistor. We know that the electric field in a perfect conductor is **zero**, and likewise in a good conductor it is **very small**. Assuming the wire is in fact made of a **good conductor** (e.g. copper), we can approximate the electric field **within** the wire (and thus at **every** point along section 1) as **zero** (i.e., $\mathbf{E}(\bar{r}) = 0$). Therefore, this first integral equals zero!

$$\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

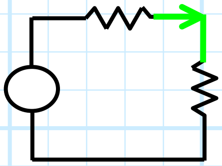
This of course makes sense! We know that the electric potential difference across a **wire** is **zero volts**.



Section 2 (*b* to *c*)

In this section, the contour moves through the first **resistor**. The contour integral along this section therefore allows us to determine the electric **potential difference** across this resistor. Let's denote this potential difference as v_1 :

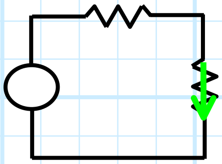
$$\int_b^c \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(P_b) - V(P_c) = v_1$$



Section 3 (*c* to *d*)

Just like section 1, the contour follows a **wire**, and thus the electric field along this section of the contour is **zero**, as is the potential difference between point *c* and point *d*.

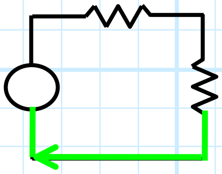
$$\int_c^d \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$



Section 4 (*d* to *e*)

Just like section 2, the contour moves through a **resistor**. The contour integral for this section is thus equal to the potential difference across this **second** resistor, which we denote as v_2:

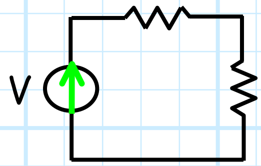
$$\int_d^e \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(P_d) - V(P_e) = v_2$$



Section 5 (*e* to *f*)

Again, the contour follows a conducting **wire**—and again, the electric field along the contour and the potential difference across it are both **zero**:

$$\int_e^f \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$



Section 6 (f to a)

This **final** section of contour C_1 extends through the **voltage source**, thus the contour integral of this section provides the electric potential difference between the two terminals of the this voltage source (i.e., $V(P_f)-V(P_a)$). By **definition**, the potential difference between points a and f is a value of V volts (i.e., $V(P_a)-V(P_f)=\text{V}$). Therefore, we find that the contour integral of section 6 is :

$$\begin{aligned}\int_f^a \mathbf{E}(\bar{r})\cdot\overline{d\ell} &= V(P_f)-V(P_a)\\ &= -\left(V(P_a)-V(P_f)\right)\\ &= -\text{V}\end{aligned}$$

Whew! Now let's **combine** these results to determine the contour integral for the **entire** contour C_1.

$$\begin{aligned}\oint_{C_1} \mathbf{E}(\bar{r})\cdot\overline{d\ell} &= \int_a^b \mathbf{E}(\bar{r})\cdot\overline{d\ell} + \int_b^c \mathbf{E}(\bar{r})\cdot\overline{d\ell} + \int_c^d \mathbf{E}(\bar{r})\cdot\overline{d\ell} +\\ &\quad \int_d^e \mathbf{E}(\bar{r})\cdot\overline{d\ell} + \int_e^f \mathbf{E}(\bar{r})\cdot\overline{d\ell} + \int_f^a \mathbf{E}(\bar{r})\cdot\overline{d\ell}\\ &= 0+v_1+0+v_2+0-V\\ &= v_1+v_2-V\end{aligned}$$

Q: *Wait; I've forgotten, Why are we evaluating these contour integrals ?*

A: Remember, since the electric field is **static**, we also know that integral around any closed contour is **zero**. Thus, we can conclude that:

$$0=\oint_{C_1} \mathbf{E}(\bar{r})\cdot\overline{d\ell}=v_1+v_2-V$$

In other words, we find by performing an **electromagnetic** analysis of the circuit, the voltages across each circuit element are related as:

$$v_1+v_2-V=0$$

Q: *You **have** wasted my time! Using **only** Kirchoff's Voltage Law (KVL), **I** arrived at **precisely** the same result ($v_1+v_2-V=0$). **I** think the above equation is true because of **KVL**, not because of your **fancy** electromagnetic theory!*

A: It **is** true that the result we obtained by integrating the electric field around the circuit contour is **likewise** apparent from **KVL**. However, this result is **still** attributable to electrostatic physics, because KVL is a **direct** result of electrostatics!

The electrostatic equation :

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

when applied to the closed contour of any **circuit**, results in **Kirchoff's Voltage Law**, i.e.:

$$\sum_n v_n = 0$$

where v_n are the electric potential differences across each element of a circuit "loop" (i.e., closed contour).

Gustav Robert Kirchhoff (1824-1887), German physicist, announced the laws that allow calculation of the currents, voltages, and resistances of electrical networks in 1845, when he was only **twenty-one**! His other work established the technique of spectrum analysis that he applied to determine the composition of the Sun.

From www.ee.umd.edu/~taylor/frame5.htm

Joule's Law

Recall that the **work** done **on** charge Q **by** an electric field in moving the charge along some **contour** C is:

$$W = Q \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell}$$

Q: *Say instead of one charge Q, we have a steady* ***stream*** *of charges (i.e., electric current) flowing along contour C?*

A: We would need to determine the **rate** of work **per unit time**, i.e., the **power** applied by the field to the current.

Recall also that the **time derivative** of work is power!

$$P = \frac{dW}{dt} = \frac{d}{dt}\left(Q \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell} \right)$$

Since the electric field is **static**, we can write:

$$\begin{aligned} P &= \frac{dQ}{dt} \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= I \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell} \end{aligned}$$

But look! The **contour integral** we know is equal to the **potential difference** V between either end of the contour. Therefore:

$$\begin{aligned} P &= I \int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} \\ &= I\,V \end{aligned}$$

Look familiar!?

The **power** delivered to charges by the field is equal to the **current** I flowing along the contour, **times** the **potential difference** (i.e., voltage V) across the contour.

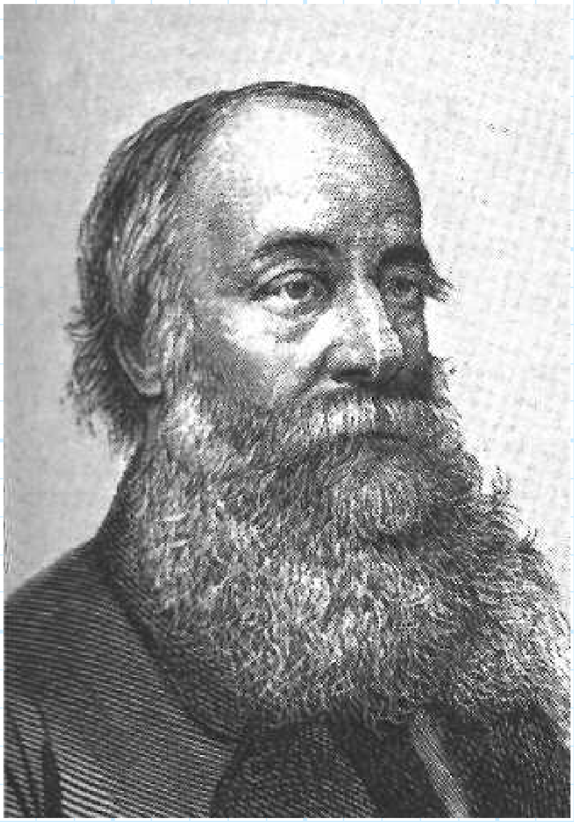
Consider now the power delivered in some **volume** V, say the volume of a resistor. Recall the electric field has units of **volts/m**, and the current density has units of **amps/m²**.

We find therefore that the **dot product** of the electric field and the current density is a **scalar** value with units of Watts/m^3. We call this scalar value the **power density**:

$$\text{power density} = \mathbf{E}(\bar{r}) \cdot \mathbf{J}(\bar{r}) \qquad \left[\left(\frac{V}{m}\right)\left(\frac{A}{m^2}\right) = \frac{W}{m^3}\right]$$

Integrating power density over some volume V gives the **total power** delivered by the field **within that volume**:

$$
\begin{aligned}
P &= \iiint_V \mathbf{E}(\bar{\mathbf{r}}) \cdot \mathbf{J}(\bar{\mathbf{r}})\, dv \\
&= \iiint_V \sigma(\bar{r}) \left|\mathbf{E}(\bar{\mathbf{r}})\right|^2 dv \qquad [\mathrm{W}] \\
&= \iiint_V \frac{1}{\sigma(\bar{r})} \left|\mathbf{J}(\bar{\mathbf{r}})\right|^2 dv
\end{aligned}
$$



James Prescott Joule (1818-1889), born into a well-to-do family prominent in the brewery industry, studied at Manchester under Dalton. At age twenty-one he published the "I-squared-R" law which bears his name. Two years later, he published the first determination of the mechanical equivalent of heat. He became a collaborator with Thomson and they discovered that the temperature of an expanding gas falls. The "Joule-Thomson effect" was the basis for the large refrigeration plants constructed in the 19th century (but not used by the British brewery industry). Joule was a patient, methodical and devoted scientist; it became known that he had taken a thermometer with him on his honeymoon and spent time attempting to measure water temperature differences at the tops and bottoms of waterfalls.

From www.ee.umd.edu/~taylor/frame5.htm