7-3 The Biot-Savart Law and the Magnetic Vector Potential

Reading Assignment: pp. 208-218

Q: Given some field $B(\vec{r})$, how can we determine the source $J(\vec{r})$ that created it?

A: Easy! $\Rightarrow J(\vec{r}) = \nabla \times B(\vec{r})/\mu_0$

Q: OK, given some source $J(\vec{r})$, how can we determine what field $B(\vec{r})$ it creates?

A:

**HO: The Magnetic Vector Potential**

**HO: Solutions to Ampere’s Law**

**HO: The Biot-Savart Law**

**Example: The Uniform, Infinite Line of Current**

**HO: B-field from an Infinite Current Sheet**
The Magnetic Vector Potential

From the magnetic form of Gauss's Law $\nabla \cdot B(\vec{r}) = 0$, it is evident that the magnetic flux density $B(\vec{r})$ is a solenoidal vector field.

Recall that a solenoidal field is the curl of some other vector field, e.g.,:

$$B(\vec{r}) = \nabla \times A(\vec{r})$$

**Q:** The magnetic flux density $B(\vec{r})$ is the curl of what vector field??

**A:** The magnetic vector potential $A(\vec{r})$!

The curl of the magnetic vector potential $A(\vec{r})$ is equal to the magnetic flux density $B(\vec{r})$:

$$\nabla \times A(\vec{r}) = B(\vec{r})$$

where:
Vector field $\mathbf{A}(\vec{r})$ is called the magnetic vector potential because of its analogous function to the electric scalar potential $\mathbf{V}(\vec{r})$.

An electric field can be determined by taking the gradient of the electric potential, just as the magnetic flux density can be determined by taking the curl of the magnetic potential:

$$\mathbf{E}(\vec{r}) = -\nabla \mathbf{V}(\vec{r}) \quad \text{and} \quad \mathbf{B}(\vec{r}) = \nabla \times \mathbf{A}(\vec{r})$$

Yikes! We have a big problem!

There are actually (infinitely) many vector fields $\mathbf{A}(\vec{r})$ whose curl will equal an arbitrary magnetic flux density $\mathbf{B}(\vec{r})$. In other words, given some vector field $\mathbf{B}(\vec{r})$, the solution $\mathbf{A}(\vec{r})$ to the differential equation $\nabla \times \mathbf{A}(\vec{r}) = \mathbf{B}(\vec{r})$ is not unique!

But of course, we knew this!

To completely (i.e., uniquely) specify a vector field, we need to specify both its divergence and its curl.
Well, we know the curl of the magnetic vector potential $\mathbf{A}(\mathbf{r})$ is equal to magnetic flux density $\mathbf{B}(\mathbf{r})$. But, what is the divergence of $\mathbf{A}(\mathbf{r})$ equal to? I.E.,:

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = ???$$

By answering this question, we are essentially defining $\mathbf{A}(\mathbf{r})$.

Let's define it in so that it makes our computations easier!

To accomplish this, we first start by writing Ampere's Law in terms of magnetic vector potential:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

We recall from section 2-6 that:

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \nabla (\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r})$$

Thus, we can simplify this statement if we decide that the divergence of the magnetic vector potential is equal to zero:

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

We call this the gauge equation for magnetic vector potential. Note the magnetic vector potential $\mathbf{A}(\mathbf{r})$ is therefore also a solenoidal vector field.
As a result of this gauge equation, we find:

\[ \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \nabla (\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = -\nabla^2 \mathbf{A}(\mathbf{r}) \]

And thus Ampere's Law becomes:

\[ \nabla \times \mathbf{B}(\mathbf{r}) = -\nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \]

Note the Laplacian operator \( \nabla^2 \) is the vector Laplacian, as it operates on vector field \( \mathbf{A}(\mathbf{r}) \).

Summarizing, we find the magnetostatic equations in terms of magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) are:

\[ \begin{align*}
\nabla \times \mathbf{A}(\mathbf{r}) &= \mathbf{B}(\mathbf{r}) \\
\nabla^2 \mathbf{A}(\mathbf{r}) &= -\mu_0 \mathbf{J}(\mathbf{r}) \\
\nabla \cdot \mathbf{A}(\mathbf{r}) &= 0
\end{align*} \]

Note that the magnetic form of Gauss's equation results in the equation \( \nabla \cdot \nabla \times \mathbf{A}(\mathbf{r}) = 0 \). Why don't we include this equation in the above list?
Compare the magnetostatic equations using the magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) to the electrostatic equations using the electric scalar potential \( \mathbf{V}(\mathbf{r}) \):

\[
\mathbf{E}(\mathbf{r}) = -\nabla \mathbf{V}(\mathbf{r})
\]

\[
\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_v(\mathbf{r})}{\varepsilon_0}
\]

Hopefully, you see that the two potentials \( \mathbf{A}(\mathbf{r}) \) and \( \mathbf{V}(\mathbf{r}) \) are in many ways analogous.

For example, we know that we can determine a static field \( \mathbf{E}(\mathbf{r}) \) created by sources \( \rho_v(\mathbf{r}) \) either directly (from Coulomb’s Law), or indirectly by first finding potential \( \mathbf{V}(\mathbf{r}) \) and then taking its derivative (i.e., \( \mathbf{E}(\mathbf{r}) = -\nabla \mathbf{V}(\mathbf{r}) \)).

Likewise, the magnetostatic equations above say that we can determine a static field \( \mathbf{B}(\mathbf{r}) \) created by sources \( \mathbf{J}(\mathbf{r}) \) either directly, or indirectly by first finding potential \( \mathbf{A}(\mathbf{r}) \) and then taking its derivative (i.e., \( \nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}) \)).

\[
\rho_v(\mathbf{r}) \Rightarrow \mathbf{V}(\mathbf{r}) \Rightarrow \mathbf{E}(\mathbf{r})
\]

\[
\mathbf{J}(\mathbf{r}) \Rightarrow \mathbf{A}(\mathbf{r}) \Rightarrow \mathbf{B}(\mathbf{r})
\]
Solutions to Ampere's Law

Say we know the current distribution $\mathbf{J}(\mathbf{r})$ occurring in some physical problem, and we wish to find the resulting magnetic flux density $\mathbf{B}(\mathbf{r})$.

**Q:** How do we find $\mathbf{B}(\mathbf{r})$ given $\mathbf{J}(\mathbf{r})$?

**A:** Two ways! We either directly solve the differential equation:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

Or we first solve this differential equation for vector field $\mathbf{A}(\mathbf{r})$:

$$-\nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

and then find $\mathbf{B}(\mathbf{r})$ by taking the curl of $\mathbf{A}(\mathbf{r})$ (i.e., $\nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r})$).

It turns out that the second option is often the easiest!

To see why, consider the vector Laplacian operator if vector field $\mathbf{A}(\mathbf{r})$ is expressed using Cartesian base vectors:

$$\nabla^2 \mathbf{A}(\mathbf{r}) = \nabla^2 A_x(\mathbf{r}) \hat{a}_x + \nabla^2 A_y(\mathbf{r}) \hat{a}_y + \nabla^2 A_z(\mathbf{r}) \hat{a}_z$$
We therefore write **Ampere’s Law** in terms of **three** separate **scalar** differential equations:

\[
\nabla^2 A_x (\vec{r}) = -\mu_0 J_x (\vec{r})
\]

\[
\nabla^2 A_y (\vec{r}) = -\mu_0 J_y (\vec{r})
\]

\[
\nabla^2 A_z (\vec{r}) = -\mu_0 J_z (\vec{r})
\]

Each of these differential equations is **easily solved**. In fact, we already **know** their solution!

Recall we had the **exact** same differential equation in electrostatics (i.e., Poisson’s equation):

\[
\nabla^2 V (\vec{r}) = \frac{-\rho (\vec{r})}{\varepsilon_0}
\]

We **know** the solution \( V (\vec{r}) \) to this differential equation is:

\[
V (\vec{r}) = \frac{1}{4\pi \varepsilon_0} \iiint_{V'} \frac{\rho (\vec{r}')}{|\vec{r} - \vec{r}'|} dV'
\]

Mathematically, Poisson’s equation is **exactly** the same as each of the three scalar differential equations at the top of the page, with these **substitutions**:

\[
V (\vec{r}) \rightarrow A_x (\vec{r}) \quad \rho (\vec{r}) \rightarrow J_x (\vec{r}) \quad \frac{1}{\varepsilon_0} \rightarrow \mu_0
\]
The solutions to the magnetic differential equation are therefore:

\[ A_x(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}_x(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \]

\[ A_y(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}_y(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \]

\[ A_z(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}_z(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \]

and since:

\[ \mathbf{A}(\mathbf{r}) = A_x(\mathbf{r}) \, \hat{\mathbf{a}}_x + A_y(\mathbf{r}) \, \hat{\mathbf{a}}_y + A_z(\mathbf{r}) \, \hat{\mathbf{a}}_z \]

and:

\[ \mathbf{J}(\mathbf{r}) = J_x(\mathbf{r}) \, \hat{\mathbf{a}}_x + J_y(\mathbf{r}) \, \hat{\mathbf{a}}_y + J_z(\mathbf{r}) \, \hat{\mathbf{a}}_z \]

we can combine these three solutions and get the vector solution to our vector differential equation:

\[ \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \]

Therefore, given current distribution \( \mathbf{J}(\mathbf{r}) \), we use the above equation to determine magnetic vector potential \( \mathbf{A}(\mathbf{r}) \). We then take the curl of this result to determine magnetic flux density \( \mathbf{B}(\mathbf{r}) \).
For surface current, the resulting magnetic vector potential is:

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iint_S \mathbf{J}_s(\mathbf{r}') \, ds'
\]

and for a current \( I \) flowing along contour \( C \), we find:

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{\mathbf{d}\ell'}{|\mathbf{r} - \mathbf{r}'|}
\]

Again, ponder the analogy between these equations involving sources and potentials and the equivalent equation from electrostatics:

\[
V(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \iiint_{V'} \rho_v(\mathbf{r}') \, dv'
\]
The Biot-Savart Law

So, we now know that given some current density, we can find the resulting magnetic vector potential \( A(\vec{r}) \):

\[
A(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, dV
\]

and then determine the resulting magnetic flux density \( B(\vec{r}) \) by taking the curl:

\[
B(\vec{r}) = \nabla \times A(\vec{r})
\]

Q: Golly, can’t we somehow combine the curl operation and the magnetic vector potential integral?

A: Yes! The result is known as the Biot-Savart Law.

Combining the two above equations, we get:

\[
B(\vec{r}) = \nabla \times \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, dV
\]

This result is of course not very helpful, but we note that we can move the curl operation into the integrand:
\[ B(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_d \nabla \times \frac{J(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\nu' \]

Note this result reverses the process: first we perform the curl, and then we integrate.

We can do this is because the integral is over the primed coordinates (i.e., \( \vec{r}' \)) that specify the sources (current density), while the curl take the derivatives of the unprimed coordinates (i.e., \( \vec{r} \)) that describe the fields (magnetic flux density).

**Q:** Yikes! That curl operation still looks particularly difficult. How we perform it?

**A:** We take advantage of a know vector identity! The curl of vector field \( f(\vec{r})G(\vec{r}) \), where \( f(\vec{r}) \) is any scalar field and \( G(\vec{r}) \) is any vector field, can be evaluated as:

\[ \nabla \times (f(\vec{r})G(\vec{r})) = f(\vec{r})\nabla \times G(\vec{r}) - G(\vec{r}) \times \nabla f(\vec{r}) \]

Note the integrand of the above equation is in the form \( \nabla \times (f(\vec{r})G(\vec{r})) \), where:

\[ f(\vec{r}) = \frac{1}{|\vec{r} - \vec{r}'|} \quad \text{and} \quad G(\vec{r}) = J(\vec{r}') \]

Therefore we find:
The Biot Savart Law.doc 3/4

\[ \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \times \mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}') \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \]

In the first term we take the curl of \( \mathbf{J}(\mathbf{r}') \). Note however that this vector field is a constant with respect to the unprimed coordinates \( \mathbf{r} \). Thus the derivatives in the curl will all be equal to zero, and we find that:

\[ \nabla \times \mathbf{J}(\mathbf{r}') = 0 \]

Likewise, it can be shown that:

\[ \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \]

Using these results, we find:

\[ \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

and therefore the magnetic flux density is:

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \]

This is know as the Biot-Savart Law!
For a surface current $\mathbf{J}_s(\mathbf{r'})$, the Biot-Savart Law becomes:

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_S \mathbf{J}_s(\mathbf{r'}) \times \frac{(\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \, ds'
$$

and for line current $I$, flowing on contour $C$, the Biot-Savart Law is:

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\ell' x (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}
$$

Note the contour $C$ is closed. Do you know why?

This is dad-gum outstanding! The Biot-Savart Law allows us to directly determine magnetic flux density $\mathbf{B}(\mathbf{r})$, given some current density $\mathbf{J}(\mathbf{r})$!

Note that the Biot-Savart Law is therefore analogous to Coulomb's Law in Electrostatics (Do you see why?)!
Example: The Uniform, Infinite Line of Current

Consider electric current $I$ flowing along the $z$-axis from $z = -\infty$ to $z = \infty$. What magnetic flux potential $B(\vec{r})$ is created by this current?

$$d\ell = \hat{z} \, dz'$$

$$\vec{r} = x \hat{x} + y \hat{y} + z \hat{z}$$

$$= \rho \cos \phi \hat{x} + \rho \sin \phi \hat{y} + z \hat{z}$$

$$\vec{r}' = z' \hat{z} \quad (x' = 0, y' = 0)$$

$$|\vec{r} - \vec{r}'| = \sqrt{\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi + (z - z')^2}$$

$$= \sqrt{\rho^2 + (z - z')^2}$$

We can determine the magnetic flux density by applying the Biot-Savart Law:
\[ B(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\ell' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \]

\[ = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \hat{a}_z \times \left[ \rho \cos \phi \hat{a}_x + \rho \sin \phi \hat{a}_y + (z - z') \hat{a}_z \right] \frac{dz'}{\left[ \rho^2 + (z - z')^2 \right]^{3/2}} \]

\[ = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho \cos \phi \hat{a}_y - \rho \sin \phi \hat{a}_x}{\left[ \rho^2 + (z - z')^2 \right]^{3/2}} \, dz' \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \cos \phi \hat{a}_y - \rho \sin \phi \hat{a}_x \right) \int_{-\infty}^{\infty} \frac{du}{\left[ \rho^2 + u^2 \right]^{3/2}} \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \hat{a}_\phi \right) \left. \right|_{-\infty}^{\infty} \frac{u}{\rho^2 \sqrt{\rho^2 + u^2}} \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \hat{a}_\phi \right) \frac{2}{\rho^2} \]

\[ = \frac{\mu_0 I}{2\pi \rho} \hat{a}_\phi \]

Therefore, the magnetic flux density created by a "wire" with current \( I \) flowing along the z-axis is:

\[ B(\vec{r}) = \frac{\mu_0 I}{2\pi \rho} \hat{a}_\phi \]
Think about what this expression tells us about magnetic flux density:

* The magnitude of $\mathbf{B}(\mathbf{r})$ is proportional to $1/\rho$, therefore magnetic flux density diminishes as we move farther from “wire”.

* The direction of $\mathbf{B}(\mathbf{r})$ is $\hat{\alpha}_\phi$. In other words, the magnetic flux density points in the direction around the wire.

Plot of vector field $\mathbf{B}(\mathbf{r})$ on the $x$-$y$ plane, resulting from current $I$ flowing along the $z$-axis

○ = current $I$ flowing out of this page.
Or, plotting in 3-D:
**B-Field from an Infinite Sheet of Current**

Consider now an infinite sheet of current, lying on the $z = 0$ plane. Say the surface current density on this sheet has a value:

$$J_s(\vec{r}) = J_x \hat{a}_x$$

meaning that the current density at every point on the surface has the same magnitude, and flows in the $\hat{a}_x$ direction.

Using the Biot-Savart Law, we find that the magnetic flux density produced by this infinite current sheet is:
Think about what this expression is telling us.

* The magnitude of this magnetic flux density is a constant. In other words, $B(\bar{r})$ is just as large a million miles from the infinite current sheet as it is 1 millimeter from the current sheet!

* The direction of the magnetic flux density in the $-\hat{a}_y$ direction above the current sheet, but points in the opposite direction (i.e., $\hat{a}_y$) below it.

* The direction of the magnetic flux density is orthogonal to the direction of current flow $\hat{a}_x$.

Plotting the vector field $B(\bar{r})$ along the $y$-$z$ plane, we find:

\[
B(\bar{r}) = \begin{cases} 
-\frac{\mu_0 J_x}{2} \hat{a}_y & z > 0 \\
\frac{\mu_0 J_x}{2} \hat{a}_y & z < 0
\end{cases}
\]