7-4 Field Calculations Using Ampere's Law

Q: Using the **Biot-Savart Law** is even more **difficult** than using Coloumb's law. Is there an **easier** way?

A:

HO: B-field from Cylindrically Symmeteric Current Distributions

Example: A Hollow Tube of Current

Example: The B-field of a Coaxial Transmission Line

HO: Solenoids

B-Field from Cylindrically Symmetric Current Distributions

Recall we discussed **cylindrically symmetric** charge distributions in Section 4-5. We found that a cylindrically symmetric charge distribution is a function of coordinate ρ **only** (i.e., $\rho_v(\bar{\mathbf{r}}) = \rho_v(\rho)$).

Similarly, we can define a cylindrically symmetric **current** distribution. A current density $\mathbf{J}(\bar{\mathbf{r}})$ is said to be cylindrically symmetric if it points in the direction $\hat{a}_z$ and is a function of coordinate ρ **only**:

$$\mathbf{J}(\bar{\mathbf{r}}) = J_z(\rho)\ \hat{a}_z$$

In other words, $J_\rho = J_\phi = 0$, and J_z is **independent** of both coordinates z and ϕ.

We find that a cylindrically symmetric current density will **always** produce a magnetic flux density of the form:

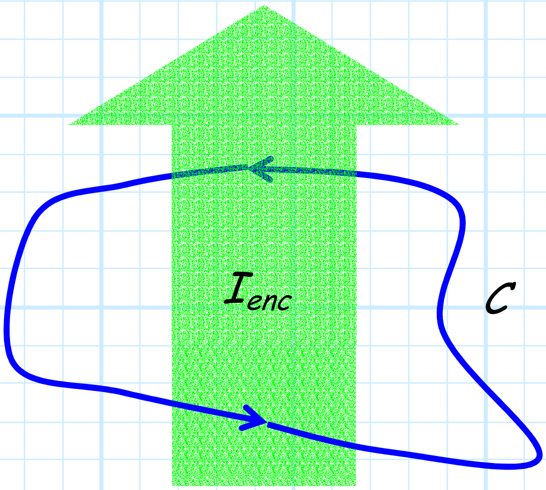
$$\mathbf{B}(\bar{r}) = B_\phi(\rho)\,\hat{a}_\phi$$

In other words, $B_\rho = B_z = 0$, and B_ϕ is independent of **both** coordinates z and ϕ.

Now, lets apply these results to the **integral** form of **Ampere's Law**:

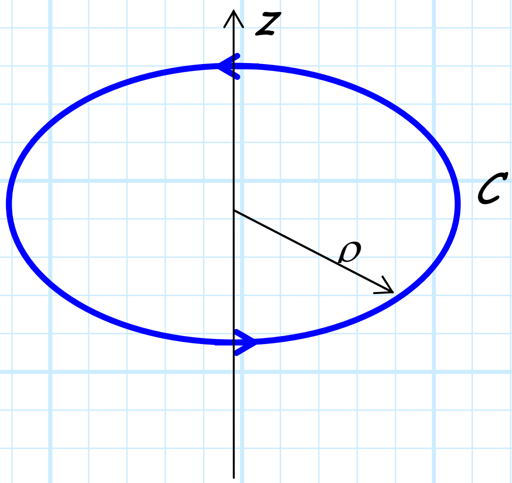
$$\oint_C \mathbf{B}(\bar{\mathbf{r}}) \cdot \overline{d\ell} = \oint_C B_\phi(\rho)\hat{a}_\phi \cdot \overline{d\ell} = \mu_0 I_{enc}$$

where you will recall that I_{enc} is the total **current** flowing **through** the aperture formed by contour C:



Say we choose for contour C a **circle**, centered around the z-axis, with radius ρ.

$$\overline{d\ell} = \hat{a}_\phi \ \rho \, d\phi$$



Amperian Path for Cylindrically Symmetric Current Distributions

This is a special contour, called the **Amperian Path** for **cylindrically symmetric** current densities. To see why it is **special**, let us use it in the cylindrically symmetric form of Ampere's Law:

$$\oint_C B_\phi(\rho)\hat{a}_\phi \cdot \overline{d\ell} = \mu_0 I_{enc}$$

$$\int_0^{2\pi} B_\phi(\rho)\hat{a}_\phi \cdot \hat{a}_\phi \, \rho \, d\phi =$$

$$B_\phi(\rho)\,\rho \int_0^{2\pi} d\phi =$$

$$2\pi\rho \; B_\phi(\rho) = \mu_0 I_{enc}$$

From this result, we can conclude that:

$$B_\phi(\rho) = \frac{\mu_0 \, I_{enc}}{2\pi\rho}$$

Q: But what is I_{enc}?

A: The current flowing **through** the circular aperture formed by contour C!

We of course can determine this by integrating the **current density** $\mathbf{J}(\overline{r})$ across the surface of this circular aperture ($\overline{ds} = \hat{a}_z \; \rho \, d\rho \, d\phi$):

$$
\begin{aligned}
I_{enc} &= \iint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds} \\
&= \int_0^{2\pi} \int_0^{\rho} J_z(\rho') \, \hat{a}_z \cdot \hat{a}_z \, \rho' \, d\rho' \, d\phi \\
&= 2\pi \int_0^{\rho} J_z(\rho') \;\; \rho' \, d\rho'
\end{aligned}
$$

Combining these results, we find that the magnetic flux density $\mathbf{B}(\bar{\mathrm{r}})$ created by a **cylindrically symmetric** current density $\mathbf{J}(\bar{\mathrm{r}})$ is:

$$
\begin{aligned}
\mathbf{B}(\bar{\mathrm{r}}) &= \frac{\mu_0 \, I_{enc}}{2\pi\rho} \; \hat{a}_\phi \\
&= \hat{a}_\phi \, \frac{\mu_0}{\rho} \int_0^{\rho} J_z(\rho') \;\; \rho' \, d\rho'
\end{aligned}
$$

For **example**, consider again a **wire** with current I flowing along the z-axis. This is a **cylindrically symmetric** current, and the total current enclosed by an **Amperian path** is clearly I for all ρ (i.e., $I_{enc} = I$).

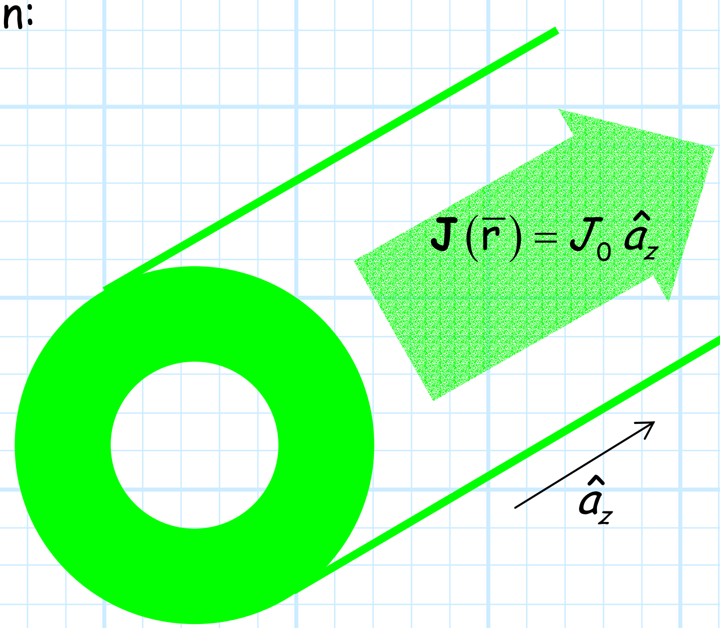
From the expression above, the magnetic flux density $\mathbf{B}(\bar{\mathrm{r}})$ is therefore:

$$
\mathbf{B}(\bar{\mathrm{r}}) = \frac{\mu_0 I}{2\pi \, \rho} \; \hat{a}_\phi
$$

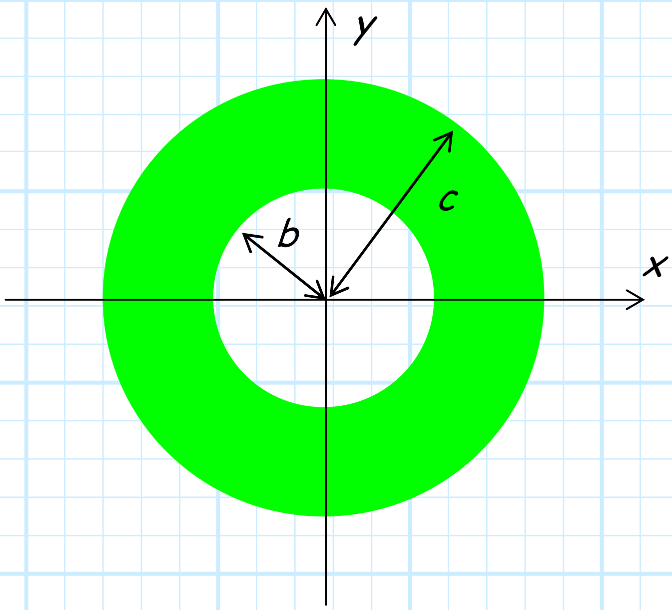
The **same** result as determined by the **Biot-Savart Law**!

Example: A Hollow Tube of Current

Consider a **hollow cylinder** of uniform current, flowing in the $\hat{a}_z$ direction:



The **inner** surface of the hollow cylinder has radius b, while the **outer** surface has radius c.



The current density in the hollow cylinder is **uniform**, thus we can express current density $\mathbf{J}(\bar{\mathbf{r}})$ as:

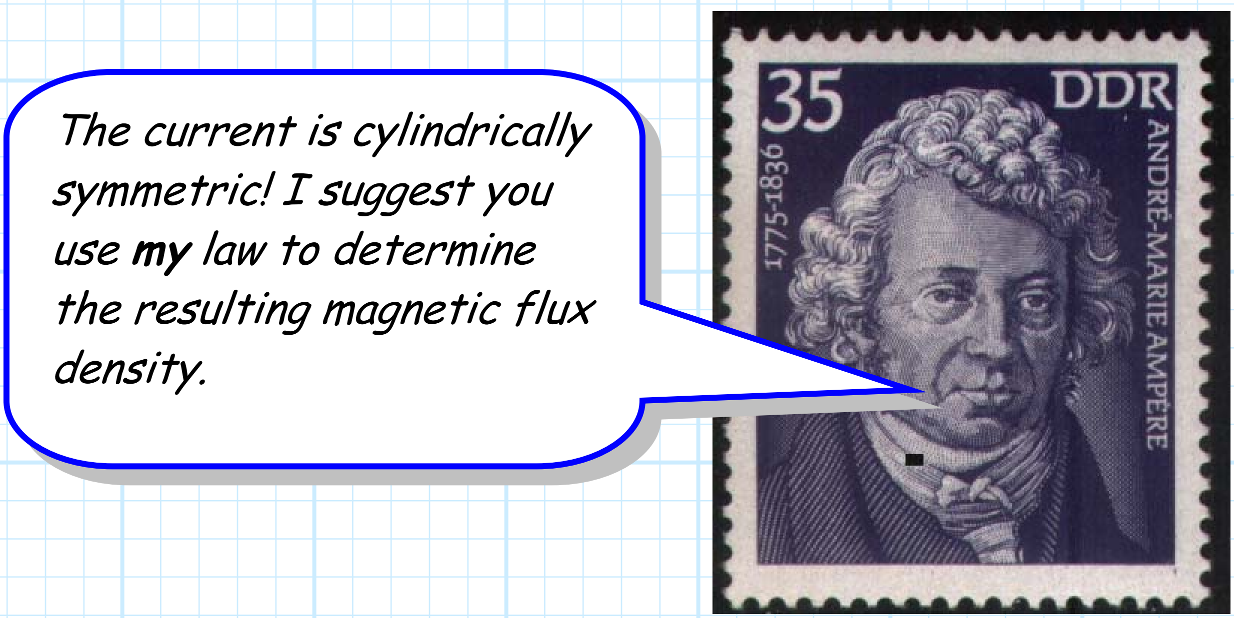
$$\mathbf{J}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ J_0 \hat{a}_z & b < \rho < c \\ 0 & \rho > c \end{cases} \qquad \left[\frac{Amps}{m^2}\right]$$

Q: *What magnetic flux density $\mathbf{B}(\bar{\mathbf{r}})$ is produced by this current density $\mathbf{J}(\bar{\mathbf{r}})$?*

A: We **could** use the Biot-Savart Law to determine $\mathbf{B}(\bar{\mathbf{r}})$, but note that $\mathbf{J}(\bar{\mathbf{r}})$ is **cylindrically symmetric**!

In other words, current density $\mathbf{J}(\bar{\mathbf{r}})$ has the form:

$$\mathbf{J}(\bar{\mathbf{r}}) = J_z(\rho)\,\hat{a}_z$$



Recall using **Ampere's Law,** we determined that **cylindrically symmetric** current densities produce magnetic flux densities of the form:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0 \, I_{enc}}{2\pi\rho} \, \hat{a}_\phi$$

$$= \hat{a}_\phi \, \frac{\mu_0}{\rho} \int_0^{\rho} J_z(\rho') \;\; \rho' \, d\rho'$$

Therefore, we must evaluate the integral for the current density in this case. Because of the piecewise nature of the current density, we must evaluate the integral for **three** different cases:

1) when the radius of the Amperian path is **less than *b*** (i.e., $\rho < b$).

2) when the radius of the Amperian path is **greater than *b*** but **less than *c*** (i.e., $b < \rho < c$).

3) when the radius of the Amperian path is **greater than *c*.**

$\underline{\rho < b}$

Note for $\rho < b$, $\mathbf{J}(\bar{r}) = 0$ and therefore the integral is **zero**:

$$\int_0^{\rho} J_z(\rho') \;\; \rho' \, d\rho' \; = \int_0^{\rho} 0 \;\; \rho' \, d\rho' = 0$$

and therefore:

$$\mathbf{B}(\bar{\mathbf{r}}) = \hat{a}_\phi \frac{\mu_0}{\rho}\, 0$$
$$= 0 \qquad\qquad \text{for} \quad \rho < b$$

Thus, the magnetic flux density in the hollow region of the cylinder is **zero**!

$b < \rho < c$

Note for $b < \rho < c$, $\mathbf{J}(\bar{\mathbf{r}}) = J_0\, \hat{a}_z$ (i.e., $J_z(\rho) = J_0$) and therefore:

$$\begin{aligned}
\int_0^\rho J_z(\rho')\ \rho' d\rho' &= \int_0^b J_z(\rho')\ \rho' d\rho' + \int_b^\rho J_z(\rho')\ \rho' d\rho' \\
&= \int_0^b 0\ \ \rho' d\rho' + \int_b^\rho J_0\ \ \rho' d\rho' \\
&= 0 + J_0 \int_b^\rho \rho' d\rho' \\
&= J_0 \left(\frac{\rho^2}{2} - \frac{b^2}{2} \right) \\
&= J_0 \left(\frac{\rho^2 - b^2}{2} \right)
\end{aligned}$$

and therefore the magnetic flux density in the non-hollow portion of the cylinder is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \hat{a}_\phi \frac{\mu_0}{\rho} \; J_0 \left(\frac{\rho^2 - b^2}{2} \right) \qquad \text{for} \quad b < \rho < c$$

$\rho > c$

Note that outside the cylinder (i.e., $\rho > c$), the current density $\mathbf{J}(\bar{\mathbf{r}})$ is again **zero**, and therefore:

$$\begin{aligned}
\int_0^\rho J_z(\rho')\,\rho'\,d\rho' &= \int_0^b J_z(\rho')\,\rho'\,d\rho' + \int_b^c J_z(\rho')\,\rho'\,d\rho' + \int_c^\rho J_z(\rho')\,\rho'\,d\rho' \\
&= \int_0^b 0 \;\; \rho'\,d\rho' + \int_b^c J_0 \;\; \rho'\,d\rho' + \int_c^\rho 0 \;\; \rho'\,d\rho' \\
&= 0 + J_0 \int_b^c \rho'\,d\rho' + 0 \\
&= J_0 \left(\frac{c^2}{2} - \frac{b^2}{2} \right) \\
&= J_0 \left(\frac{c^2 - b^2}{2} \right)
\end{aligned}$$

Thus, the magnetic flux density **outside** the current cylinder is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \hat{a}_\phi \frac{\mu_0}{\rho} \; J_0 \left(\frac{c^2 - b^2}{2} \right) \qquad \text{for} \quad c > \rho$$

Summarizing, we find that the **magnetic flux density** produced by this hollow tube of current is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ \dfrac{J_0\ \mu_0}{\rho}\left(\dfrac{\rho^2 - b^2}{2}\right)\hat{a}_\phi & b < \rho < c \\ \dfrac{J_0\ \mu_0}{\rho}\left(\dfrac{c^2 - b^2}{2}\right)\hat{a}_\phi & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

We can find an **alternative** expression by determining the total **current** flowing through this cylinder (let's call this current I_0). We of course can determine I_0 by performing the **surface integral** of the current density $\mathbf{J}(\bar{\mathbf{r}})$ across the cross sectional surface S of the cylinder:

$$\begin{aligned} I_0 &= \iint_S \mathbf{J}(\bar{\mathbf{r}}) \cdot \overline{ds} \\ &= \int_0^{2\pi}\int_b^c J_0\ \hat{a}_z \cdot \hat{a}_z\ \rho\, d\rho\, d\phi \\ &= J_0 \int_0^{2\pi}\int_b^c \rho\, d\rho\, d\phi \\ &= J_0\ \pi\left(c^2 - b^2\right) \end{aligned}$$

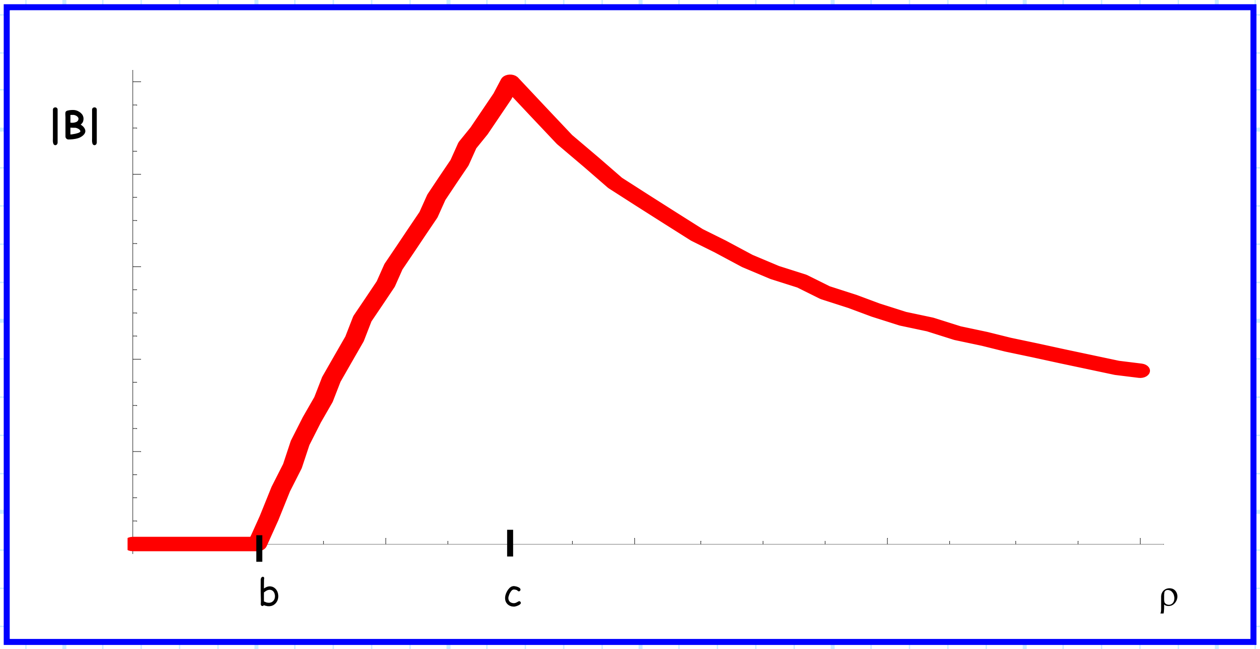
Therefore, we can conclude that:

$$J_0 = \frac{I_0}{\pi\left(c^2 - b^2\right)}$$

Inserting this into the expression for the magnetic flux density, we find:

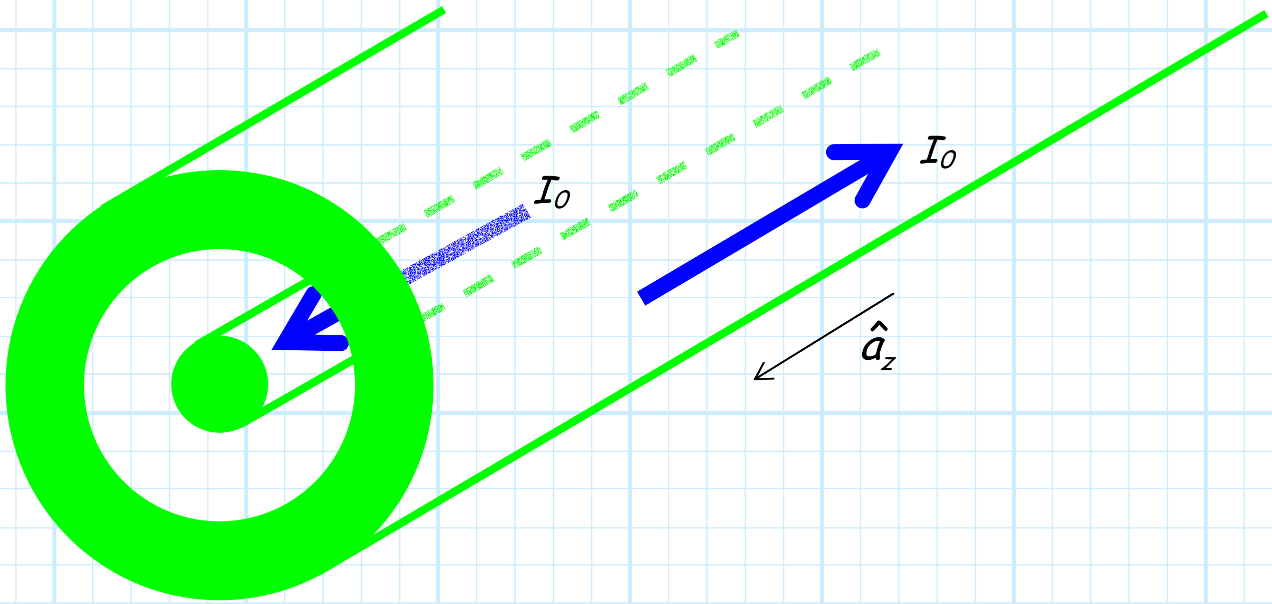
$$\mathbf{B}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ \dfrac{I_0\ \mu_0}{2\pi\rho}\left(\dfrac{\rho^2 - b^2}{c^2 - b^2}\right)\hat{a}_\phi & b < \rho < c \\ \dfrac{I_0\ \mu_0}{2\pi\rho}\ \hat{a}_\phi & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

Note the field outside of the cylinder ($\rho > c$) behaves precisely as would the field from a wire of current I_0!

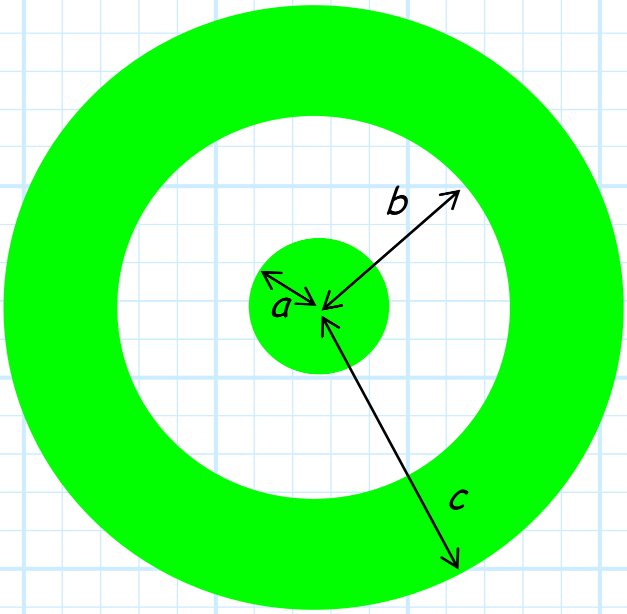


Example: The B-Field of Coaxial Transmission Line

Consider now a **coaxial cable**, with inner radius a :



The **outer** surface of the **inner** conductor has radius a, the **inner** surface of the **outer** conductor has radius b, and the **outer** radius of the **outer** conductor has radius c.



Typically, the **current** flowing on the **inner** conductor is **equal but opposite** that flowing in the **outer** conductor. Thus, if current I_0 is flowing in the **inner** conductor in the direction $\hat{a}_z$, then current I_0 will be flowing in the **outer** conductor in the opposite (i.e., $-\hat{a}_z$) direction.

Q: *Hey! If there is current, a magnetic flux density must be created. What is the vector field* $\mathbf{B}(\bar{r})$*?*

A: We've already determined this (sort of) !

Recall we found the magnetic flux density produced by a hollow cylinder—we can use this to determine the magnetic flux density in a coaxial transmission line.

→ A coaxial cable can be viewed as **two** hollow cylinders!

Q: ***I*** *find it necessary to point out that you are indeed* ***wrong***—*the* ***inner*** *conductor is* ***not hollow***!

A: Mathematically, we can view the inner conductor as a hollow cylinder with an outer radius *a* and an **inner** radius of **zero**!

Thus, we can use the results of the previous handout to conclude that the magnetic flux density produced by the current flowing in the inner conductor is:

$$\mathbf{B}_{inner}(\bar{\mathbf{r}}) = \begin{cases} \dfrac{I_0\ \mu_0}{2\pi\rho}\left(\dfrac{\rho^2 - 0^2}{a^2 - 0^2}\right)\hat{a}_\phi = \dfrac{I_0\ \mu_0}{2\pi a^2}\rho\ \hat{a}_\phi & \rho < a \\ \\ \dfrac{I_0\ \mu_0}{2\pi\rho}\hat{a}_\phi & \rho > a \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

Likewise, we can use the same result to determine the magnetic flux density of the current flowing in the outer conductor:

$$\mathbf{B}_{outer}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ \\ \dfrac{-I_0\ \mu_0}{2\pi\rho}\left(\dfrac{\rho^2 - b^2}{c^2 - b^2}\right)\hat{a}_\phi & b < \rho < c \\ \\ \dfrac{-I_0\ \mu_0}{2\pi\rho}\hat{a}_\phi & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

Note the **minus sign** is due to direction of the current $(-\hat{a}_z)$ in the outer conductor.

We can now apply **superposition** to determine the total magnetic flux density in a coaxial transmission line! Specifically:

$$\text{if} \qquad \mathbf{J}(\bar{r}) = \mathbf{J}_{inner}(\bar{r}) + \mathbf{J}_{outer}(\bar{r})$$

$$\text{then} \qquad \mathbf{B}(\bar{r}) = \mathbf{B}_{inner}(\bar{r}) + \mathbf{B}_{outer}(\bar{r})$$

Note due to the **piecewise** nature of these solutions, we must evaluate this sum for **4** distinct regions:

1) $\rho < a$ (in the inner conductor)

2) $a < \rho < b$ (in the region between the conductors)

3) $b < \rho < c$ (in the outer conductor)

4) $\rho > c$ (outside the coaxial cable)

$\underline{\rho < a}$

$$\begin{aligned} \mathbf{B}(\bar{r}) &= \mathbf{B}_{inner}(\bar{r}) + \mathbf{B}_{outer}(\bar{r}) \\ &= \frac{I_0\,\mu_0}{2\pi a^2}\,\rho\,\hat{a}_\phi + 0 \\ &= \frac{I_0\,\mu_0}{2\pi a^2}\,\rho\,\hat{a}_\phi \end{aligned}$$

$\underline{a < \rho < b}$

$$\begin{aligned}\mathbf{B}(\bar{\mathbf{r}}) &= \mathbf{B}_{inner}(\bar{\mathbf{r}}) + \mathbf{B}_{outer}(\bar{\mathbf{r}}) \\ &= \frac{I_0\ \mu_0}{2\pi\ \rho}\ \hat{a}_\phi + 0 \\ &= \frac{I_0\ \mu_0}{2\pi\ \rho}\ \hat{a}_\phi\end{aligned}$$

$\underline{b < \rho < c}$

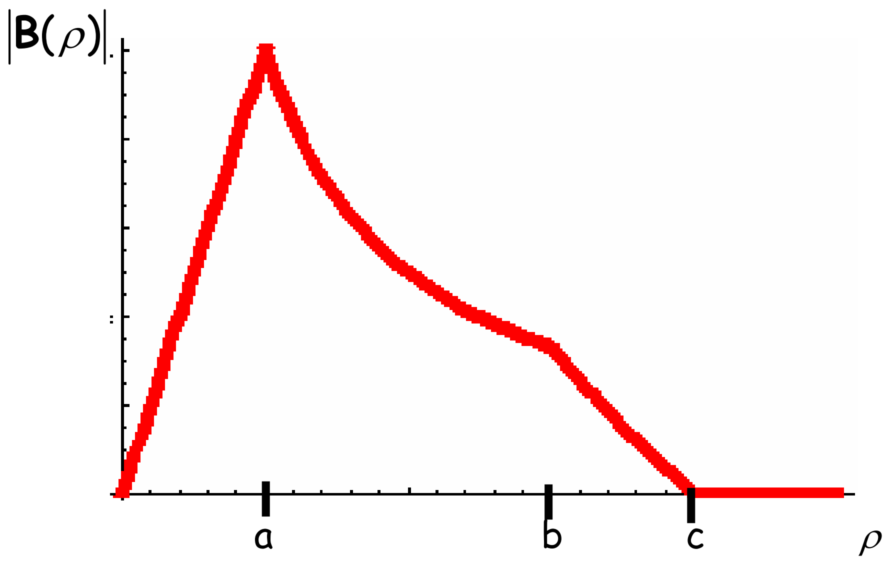
$$\begin{aligned}\mathbf{B}(\bar{\mathbf{r}}) &= \mathbf{B}_{inner}(\bar{\mathbf{r}}) + \mathbf{B}_{outer}(\bar{\mathbf{r}}) \\ &= \frac{I_0\ \mu_0}{2\pi\ \rho}\ \hat{a}_\phi + \frac{-I_0\ \mu_0}{2\pi\rho}\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)\hat{a}_\phi \\ &= \frac{I_0\ \mu_0}{2\pi\ \rho}\left(\frac{c^2 - \rho^2}{c^2 - b^2}\right)\hat{a}_\phi\end{aligned}$$

$\underline{\rho > c}$

$$\begin{aligned}\mathbf{B}(\bar{\mathbf{r}}) &= \mathbf{B}_{inner}(\bar{\mathbf{r}}) + \mathbf{B}_{outer}(\bar{\mathbf{r}}) \\ &= \frac{I_0\ \mu_0}{2\pi\ \rho}\ \hat{a}_\phi + \frac{-I_0\ \mu_0}{2\pi\rho}\ \hat{a}_\phi \\ &= 0\end{aligned}$$

Summarizing, we find the total magnetic flux density to be:

$$\mathbf{B}(\bar{\mathbf{r}}) = \begin{cases} \dfrac{I_0\ \mu_0}{2\pi a^2}\rho\ \hat{a}_\phi & \rho < a \\[2ex] \dfrac{I_0\ \mu_0}{2\pi\ \rho}\ \hat{a}_\phi & a < \rho < b \\[2ex] \dfrac{I_0\ \mu_0}{2\pi\ \rho}\left(\dfrac{c^2-\rho^2}{c^2-b^2}\right)\hat{a}_\phi & b < \rho < c \\[2ex] 0 & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

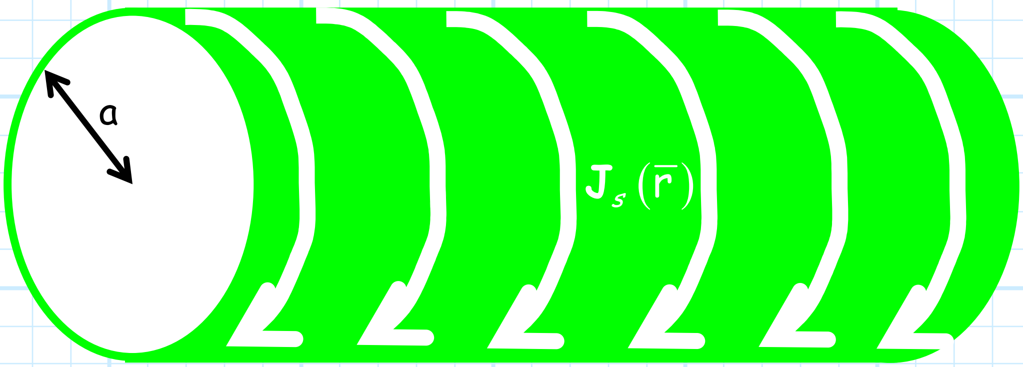


The magnetic flux density **and** the electric field **outside** of a coaxial transmission line are **zero**!

Solenoids

An important structure in electrical and computer engineering is the **solenoid**.

A solenoid is a **tube of current**. However, it is different from the hollow cylinder example, in that the current flows **around** the tube, rather than down the tube:



Aligning the center of the tube with the *z*-axis, we can express the **current density** as:

$$\mathbf{J}_s(\bar{r}) = \begin{cases} 0 & \rho < a \\ J_s\ \hat{a}_\phi & \rho = a \\ 0 & \rho > a \end{cases} \qquad \left[\frac{Amps}{m}\right]$$

where a is the **radius** of the solenoid, and J_s is the **surface** current density in Amps/meter.

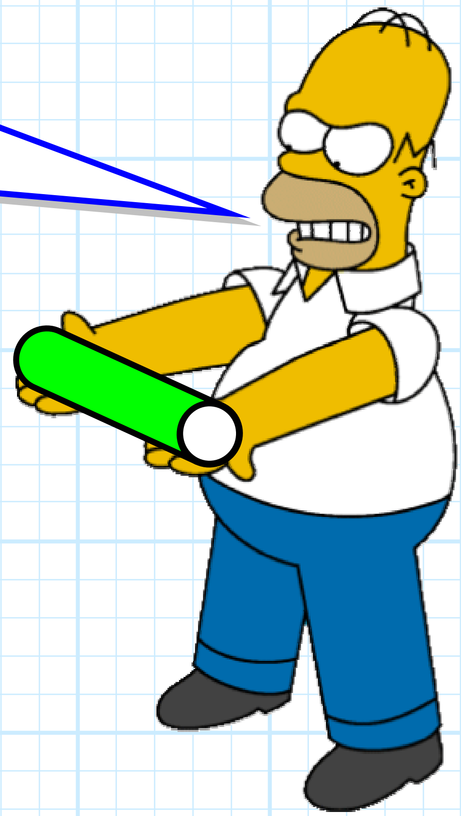
We can use **Ampere's Law** to find the magnetic flux density resulting from this structure. The result is:

$$\mathbf{B}(\bar{r}) = \begin{cases} \mu_0 J_s \, \hat{a}_z & \rho < a \\ 0 & \rho > a \end{cases}$$

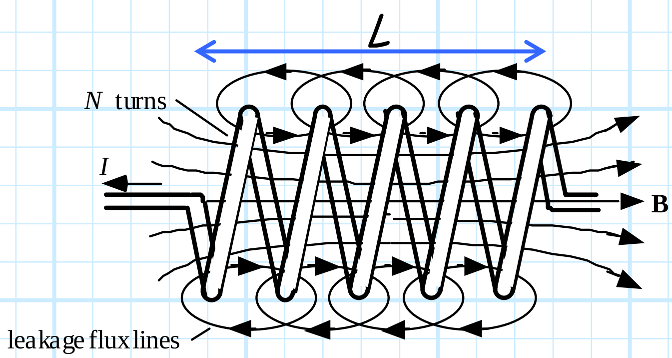
Note the direction of the magnetic flux density is in the direction $\hat{a}_z$--it points **down** the center of the solenoid.

Note also that the magnitude $|\mathbf{B}(\bar{r})|$ is **independent** of solenoid radius a!

Q: *Yeah right! How are we supposed to get current to flow* ***around*** *this tube? I don't see how this is even possible.*



A: We can easily make a solenoid by forming a **wire spiral** around a cylinder.



The surface current density J_s of this solenoid is **approximately** equal to:

$$J_s = \frac{N\ I}{L} = N_\ell\ I$$

where $N_\ell = N/L$ is the number of turns/unit length. Inserting this result into our expression for magnetic flux density, we find the magnetic flux density **inside** a solenoid:

$$\begin{aligned}\mathbf{B}(\bar{r}) &= \mu_0 \frac{N\ I}{L}\ \hat{a}_z \\ &= \mu_0 N_\ell\ I\ \hat{a}_z\end{aligned}$$