7-5 Magnetic Potentials

Recall that the definition of electric scalar potential:

\[ E(\mathbf{r}) = -\nabla V(\mathbf{r}) \]

led to the integral relationship:

\[ \int_C E(\mathbf{r}) \cdot d\mathbf{r} = V(\mathbf{r}_a) - V(\mathbf{r}_b) \]

Q:

A:

HO: The Integral Definition of Magnetic Vector Potential

There are numerous analogies between electrostatics and magnetostatics!
Q:

A:

**HO: The Magnetic Dipole**
The Integral Definition of Magnetic Vector Potential

Recall for electrostatics, we began with the definition of electric scalar potential:

\[ E(\vec{r}) = -\nabla V(\vec{r}) \]

And then taking a contour integral of each side we discovered:

\[ \oint_{C} E(\vec{r}) \cdot d\ell = -\oint_{C} \nabla V(\vec{r}) \cdot d\ell \]
\[ \oint_{C} E(\vec{r}) \cdot d\ell = V(\vec{r}_{a}) - V(\vec{r}_{b}) \]

We can perform an analogous procedure for magnetic vector potential! Recall magnetic flux density \( B(\vec{r}) \) can be written in terms of the magnetic vector potential \( A(\vec{r}) \):

\[ B(\vec{r}) = \nabla \times A(\vec{r}) \]

Say we integrate both sides over some surface \( S \):

\[ \iint_{S} B(\vec{r}) \cdot d\vec{s} = \iint_{S} \nabla \times A(\vec{r}) \cdot d\vec{s} \]
We can apply **Stoke's theorem** to write the right side as:

\[
\iint_S \nabla \times \mathbf{A}(\vec{r}) \cdot d\mathbf{s} = \oint_C \mathbf{A}(\vec{r}) \cdot d\mathbf{l}
\]

Therefore, we find that we can also define magnetic vector potential in an **integral form** as:

\[
\iint_S \mathbf{B}(\vec{r}) \cdot d\mathbf{s} = \oint_C \mathbf{A}(\vec{r}) \cdot d\mathbf{l}
\]

where contour \(C\) defines the **border** of surface \(S\).

Consider now the **meaning** of the integral:

\[
\iint_S \mathbf{B}(\vec{r}) \cdot d\mathbf{s}
\]

This integral is remarkably **similar** to:

\[
\iint_S \mathbf{J}(\vec{r}) \cdot d\mathbf{s}
\]

where:

\[
\mathbf{B}(\vec{r}) \equiv \text{magnetic flux density} \quad \left[ \frac{\text{Webers}}{\text{meters}^2} \right]
\]
and:

\[ J(\vec{r}) = \text{current density} \quad \left[ \frac{\text{Amperes}}{\text{meters}^2} \right] \]

Recall that integrating the current density (in \(\text{amps/m}^2\)) over some surface \(S\) (in \(m^2\)), provided us the total current \(I\) flowing through surface \(S\):

\[ \iint_S J(\vec{r}) \cdot d\vec{s} = I \]

Similarly, integrating the magnetic flux density (in \(\text{webers/m}^2\)) over some surface \(S\) (in \(m^2\)), provided us the total magnetic flux \(\Phi\) flowing through surface \(S\):

\[ \iint_S B(\vec{r}) \cdot d\vec{s} = \Phi \]

where \(\Phi\) is defined as:

\[ \Phi = \text{magnetic flux} \quad [\text{Webers}] \]
Using the equations derived previously, we can directly relate magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) to magnetic flux as:

\[
\Phi = \oint \mathbf{A}(\mathbf{r}) \cdot d\mathbf{\ell}
\]

where we recall that the units for magnetic vector potential are Weber/m.

Note the similarities of the above expression to the integral form of Ampere’s Law!

\[
I = \frac{1}{\mu_0} \oint \mathbf{B}(\mathbf{r}) \cdot d\mathbf{\ell}
\]
The Magnetic Dipole

Consider a very small, circular current loop of radius $a$, carrying current $I$.

**Q:** What magnetic flux density is produced by this loop, in regions far from the current (i.e., $r \gg a$)?

**A:** First, find the magnetic vector potential $A(\vec{r})$:

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I \, d\ell'}{|\vec{r} - \vec{r}'|}$$

Since the contour $C$ is a circle around the $z$-axis, with radius $a$, we use the differential line vector:

$$d\ell' = \rho' d\phi' \, \hat{\phi}$$

$$= a \, d\phi' \, \hat{\phi}$$

$$= (a \cos \phi' \, \hat{x} + a \sin \phi' \, \hat{y}) \, d\phi'$$
The location of the current is specified by position vector $\vec{r}'$. Since for every point on the current loop we find $z' = 0$ and $\rho' = a$, we find:

$$\vec{r}' = x' \hat{a}_x + y' \hat{a}_y + z' \hat{a}_z$$

$$= \rho' \cos \phi' \hat{a}_x + \rho' \sin \phi' \hat{a}_y + z' \hat{a}_z$$

$$= a \cos \phi' \hat{a}_x + a \sin \phi' \hat{a}_y$$

And finally,

$$\vec{r} = x \hat{a}_x + y \hat{a}_y + z \hat{a}_z$$

$$= r \sin \theta \cos \phi \hat{a}_x + r \sin \theta \sin \phi \hat{a}_y + r \cos \theta \hat{a}_z$$

With a little algebra and trigonometry, we find also that:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \left[ r^2 - a(2r \sin \theta \cos(\phi - \phi')) + a^2 \right]^{-\frac{1}{2}}$$

Since the radius of the circle is very small (i.e., $a \ll r$), we can use a Taylor Series to approximate the above expression (see page 231 of text):

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r} + \frac{a \sin \theta \cos(\phi - \phi')}{r^2}$$

The magnetic vector potential can now be evaluated!
\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I \, d\ell'}{|\mathbf{r} - \mathbf{r}'|}
\]
\[
= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \left( \frac{1}{r} + \frac{a \sin \theta \cos (\phi - \phi')}{{r'}^2} \right) \left( a \cos \phi' \hat{\mathbf{a}}_x + a \sin \phi' \hat{\mathbf{a}}_y \right) d\phi'
\]
\[
= \frac{\pi a^2 I}{r^2} \sin \theta \left( -\sin \phi \hat{\mathbf{a}}_x + \cos \phi \hat{\mathbf{a}}_y \right)
\]
\[
= \frac{\pi a^2 I}{r^2} \sin \theta \hat{\mathbf{a}}_\phi
\]

Note that \( \pi a^2 \) equals the surface area \( S \) of the circular loop. Therefore, we can write that magnetic vector potential produced by a very small current loop is:

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0 S I}{4\pi r^2} \sin \theta \hat{\mathbf{a}}_\phi \quad (a \ll r)
\]

We can now determine magnetic flux density \( \mathbf{B}(\mathbf{r}) \) by taking the curl:

\[
\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})
\]
\[
= \frac{\mu_0 S I}{4\pi r^3} \left( 2 \cos \theta \hat{\mathbf{a}}_r + \sin \theta \hat{\mathbf{a}}_\theta \right)
\]
Q: Hey! Something about this result looks very familiar!

A: Compare this result to that of an electric dipole:

\[ E(\vec{r}) = \frac{Qd}{4\pi\varepsilon_0 r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) \]

Both results have exactly the same form:

\[ c \left( \frac{2\cos\theta \hat{r} + \sin\theta \hat{\theta}}{4\pi r^3} \right) \]

where \( c \) is a constant.
Because of this similarity, we can refer to a small current loop of area $S$ and current $I$ as a **Magnetic Dipole**.

Note that the only difference between the mathematical description of an **electric** field produced by an **electric** dipole and the **magnetic** flux density produced by a **magnetic** dipole is a constant $c$:

- Electric dipole $\rightarrow c = \frac{Q_d}{\varepsilon}$
- Magnetic dipole $\rightarrow c = \mu_0 S I$

Recall that we defined a **dipole moment** for electric dipoles, where:

$$|\mathbf{p}| = Qd$$

Clearly, the **analogous** product to $Qd$ for a magnetic dipole is $SI$. We can, in fact, define a **magnetic dipole moment** $\mathbf{m}$:

$$\mathbf{m} = \text{Magnetic Dipole Moment} \quad [\text{Amps} \cdot \text{m}^2]$$

Analogous to the electric dipole, the magnetic dipole moment has **magnitude**:

$$|\mathbf{m}| = SI$$
Q: We now know the magnitude of the magnetic dipole moment, but what is its direction??

A: The magnetic dipole \( \mathbf{m} \) points in the direction \( \text{orthogonal} \) to the circular surface \( S \), e.g.:

Note the direction is defined using the \( \text{right-hand rule} \) with respect to the direction of current \( I \).

Instead of plus (+) and minus (-), the \( \text{poles} \) of a magnetic dipole are defined as \text{north} (N) and \text{south} (S):
Thus, for the example provided on this handout, the magnetic dipole moment is:

\[ m = SI \hat{z} \]

We note that

\[ SI \sin \theta \hat{\phi} = SI \hat{z} \times \hat{r} = m \times \hat{r}, \]

therefore we can write:

\[ A(\vec{r}) = \frac{\mu_0 SI}{4\pi r^2} \sin \theta \hat{\phi}, \]

\[ = \frac{\mu_0 m \times \hat{r}}{4\pi r^2} \]

The above equation is in fact valid for any magnetic dipole located at the origin, regardless of its direction! In other words, we can also use the above expression if \( m \) is pointed in some direction other than \( \hat{z} \), e.g.:

![Diagram showing a magnetic dipole not at the origin](attachment:diagram.png)

**Q:** What if the magnetic dipole is not located at the origin?

**A:** Just like we have many times before, we make the substitutions:

\[ r \rightarrow |\vec{r} - \vec{r}'| \]

\[ \hat{a}_r = \hat{a}_{r'} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \]
Therefore, we find the magnetic flux density $\mathbf{A}(\vec{r})$ produced by an arbitrary magnetic dipole $\mathbf{m}$, located at an arbitrary position $\vec{r}'$, is:

$$
\mathbf{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}
$$

To determine the magnetic flux density $\mathbf{B}(\vec{r})$, we simply take the curl of the above expression.

Note this is analogous to the expression of the electric scalar potential generated by an electric dipole with moment $\mathbf{p}$:

$$
\mathcal{V}(\vec{r}) = \frac{1}{4\pi\varepsilon} \frac{\mathbf{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}
$$

and then taking the gradient of this function to determine the electric field $\mathbf{E}(\vec{r})$. 