8-3 Magnetic Materials

Reading Assignment: pp. 244 - 260

Recall in dielectrics, electric dipoles were created when and E-field was applied.

→ Therefore, we defined permittivity \( \varepsilon \), electric flux density \( \mathbf{D}(\mathbf{r}) \), and a new set of electrostatic equations.

Q:

A:

8-3-1 Orbital and Spin Currents

HO: Magnetic Materials

HO: The Magnetic Dipole in a B-field

8-3-2 Magnetic Susceptibility and Magnetization Currents

HO: The Magnetization Vector
**HO: Magnetization Currents**

8-3-3 The Magnetic Field Intensity

**HO: The Magnetic Field**

**Example: Magnetization Currents**

8-3-4 The Physical Properties of Magnetic Materials

**HO: Permanent Magnets**

8-3-5 Field Equations in Magnetic Materials

**HO: Field Equations in Magnetic Materials**

8-3-6 Magnetic Field Boundary Conditions

**HO: Magnetic Boundary Conditions**
Magnetic Materials

Recall that atoms and molecules, having both positive (i.e., protons) and negative (i.e., electron) charged particles can form electric dipoles.

We find that atoms and molecules also can also form magnetic dipoles!

Q: How??

A: Recall a magnetic dipole is formed when current flows in a small loop. Current, of course, is moving charge, therefore charge moving around a small loop forms a magnetic dipole.

Molecules and atoms often exhibit electrons moving around in small loops!

Again, we use our ridiculously simple model of an atom:

\[ + \quad - \quad = \text{electron (negative charge)} \]

\[ + \quad = \text{nucleus (positive charge)} \]
An electron with charge $Q$ orbiting around a nucleus at velocity $u$ forms a small current loop, where $I = Q|u|$.

This forms a **magnetic dipole**!

This is a *very simple* atomic explanation of how magnetic dipoles are formed in material. In actuality, the physical mechanisms that lead to magnetic dipoles can be *far* more complex. For example, *electron spin* can also create a magnetic dipole moment.

Typically, the atoms/molecules of materials exhibit either no magnetic dipole moment (i.e., $m = 0$), or the dipole moments of each atom/molecule are **randomly oriented**, such that the net dipole moment is **zero**.
Therefore, if we have $N$ randomly oriented magnetic dipoles $m_n$, we find the average value will be zero:

$$\frac{1}{N} \sum_n m_n = 0$$

Similarly, we find that the total magnetic flux density created by these magnetic dipoles is also zero:

$$\sum_n B_n(\vec{r}) = 0$$

However, we find that sometimes the magnetic dipole moment of each atom/molecule is not randomly oriented, but in fact are aligned!

In this case, total magnetic flux density created by these dipoles is non-zero:

$$\sum_n B_n(\vec{r}) \neq 0.$$
Q: Why would these magnetic dipoles be aligned?

A: Two possible reasons:

1) the material is a permanent magnet.

2) the material is immersed in some magnetizing field $B_m(\vec{r})$. 
The Magnetic Dipole in a B-field

Consider the case of an arbitrarily aligned magnetic dipole:

Say this dipole is immersed in some field $B_m(\vec{r})$: 
Q: What happens to a magnetic dipole when exposed to a magnetic flux density $B_m(\vec{r})$?

A: Exactly what the Lorentz Force equation says will happen!

Recall that the force $dF$ on some current element $I \, d\ell$ is:

$$dF = I \, d\ell \times B_m(\vec{r})$$

Note this force is therefore perpendicular to both $B(\vec{r})$ and current $I$.

The total resultant force on a current loop is will be zero, so the dipole does not change position. I.E.:

$$\oint_C I \, d\ell \times B_m(\vec{r}) = 0$$
However, the forces on the current do apply a **torque** $T_m$ to the current loop!

The current loop (i.e., magnetic dipole) will **rotate** until the dipole moment $\mathbf{m}$ is aligned with the magnetic flux density vector $\mathbf{B}_m(\mathbf{r})$.

For a **circular** current loop, it can be shown (pp. 234–235) that the torque applied is:

$$
T_m = \mathbf{m} \times \mathbf{B}(\mathbf{r}) \\
\text{[N} \cdot \text{m]}
$$

Note that once the magnetic dipole moment $\mathbf{m}$ is aligned with magnetic flux density $\mathbf{B}(\mathbf{r})$, the torque $T_m$ is equal to zero—the magnetic dipole stops rotating and remains aligned with $\mathbf{B}(\mathbf{r})$. 
The Magnetization Vector

Recall that we defined the Polarization vector of a dielectric material as the electric dipole density, i.e.:

\[ \mathbf{p}(\mathbf{r}) = \lim_{\Delta \nu \to 0} \frac{1}{\Delta \nu} \sum_{\nu} \mathbf{p} \left[ \text{electric dipole moment} \right] \left[ \text{unit volume} \right] \]

Similarly, we can define a Magnetization vector \( \mathbf{M}(\mathbf{r}) \) of a material to be the density of magnetic dipole moments at location \( \mathbf{r} \):

\[ \mathbf{M}(\mathbf{r}) = \lim_{\Delta \nu \to 0} \frac{1}{\Delta \nu} \sum_{\nu} \mathbf{m} \left[ \text{magnetic dipole moment} \right] \left[ \text{unit volume} \right] = \frac{A}{m} \]

Note if the dipole moments of atoms/molecules within a material are completely random, the Magnetization vector will be zero (i.e., \( \mathbf{M}(\mathbf{r}) = 0 \)).

However, if the dipoles are aligned, the Magnetization vector will be non-zero (i.e., \( \mathbf{M}(\mathbf{r}) \neq 0 \))
Recall a magnetic dipole will create a magnetic vector potential equal to:

\[ A(\vec{r}) = \frac{\mu_0 \mathbf{m} \times (\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3} \]

Since the magnetic dipole moment of some small (i.e., differential) volume \( dv \) of the material is:

\[ \mathbf{m} = \mathbf{M}(\vec{r}) dv \]

we find that the magnetic vector potential created by a volume \( V \) of material with magnetization vector \( \mathbf{M}(\vec{r}) \) is:

\[ A(\vec{r}) = \iiint_V \frac{\mu_0 \mathbf{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3} \, dv' \]

**Q:** This is freaking me out!! I thought that currents \( \mathbf{J}(\vec{r}) \) were responsible for creating magnetic vector potential. In fact, I could have sworn that:

\[ A(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_V \mathbf{J}(\vec{r}') \frac{dv'}{|\vec{r} - \vec{r}'|} \]

**A:** Relax, both expressions are correct!
Recall that we could attribute the electric field created by Polarization Vector $P(\vec{r})$ to polarization (i.e., bound) charges $\rho_{vp}(\vec{r})$ and $\rho_{sp}(\vec{r})$, i.e.,:

$$\rho_{vp}(\vec{r}) = -\nabla \cdot P(\vec{r}) \quad \rho_{sp}(\vec{r}) = P(\vec{r}) \cdot \hat{a}_n$$

Similarly, we can attribute the magnetic vector potential (and therefore the magnetic flux density) created by Magnetization Vector $M(\vec{r})$ to Magnetization Currents $J_m(\vec{r})$ and $J_{sm}(\vec{r})$. 
Magnetization Currents

Recall that the magnetic vector potential \( A(\mathbf{r}) \) created by volume current distribution \( \mathbf{J}(\mathbf{r}) \) is:

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}'
\]

while the magnetic vector potential created by a surface current \( \mathbf{J}_s(\mathbf{r}) \):

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iint_{\mathcal{S}} \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, ds'
\]

Therefore, if both volume and surface current densities are present we find that the total magnetic vector potential is:

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}' + \frac{\mu_0}{4\pi} \iint_{\mathcal{S}} \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, ds'
\]

Compare these expressions to the magnetic vector potential field produced by material with Magnetization Vector \( \mathbf{M}(\mathbf{r}) \):

\[
A(\mathbf{r}) = \iiint_{\mathcal{V}} \frac{\mu_0 \mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}'
\]

We can write also write this expression as (trust me!):
\[ A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' + \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{M}(\mathbf{r}') \times \hat{n}}{|\mathbf{r} - \mathbf{r}'|} \, ds' \]

where surface \( S \) is the **closed surface** that surrounds material volume \( V \), and unit vector \( \hat{n} \) is **normal** to this surface.

We find that this is identical to the expression:

\[ A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' + \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, ds' \]

if \( \mathbf{J}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) \) and \( \mathbf{J}_s(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{n} \).

Therefore, we find that the magnetization of some material, as described by magnetization vector \( \mathbf{M}(\mathbf{r}) \), creates **effective** currents \( \mathbf{J}_m(\mathbf{r}) \) and \( \mathbf{J}_{sm}(\mathbf{r}_s) \) (where \( \mathbf{r}_s \) indicates points on the material surface). We call these effective currents **magnetization currents**:

\[
\mathbf{J}_m(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) \quad \begin{bmatrix} A \\ m^2 \end{bmatrix}
\]

\[
\mathbf{J}_{sm}(\mathbf{r}_s) = \mathbf{M}(\mathbf{r}_s) \times \hat{n} \quad \begin{bmatrix} A \\ m \end{bmatrix}
\]

Again, note the **analogy** of these magnetization currents with **polarization** charges \( \rho_{vp}(\mathbf{r}) \) and \( \rho_{sp}(\mathbf{r}) \).
The Magnetic Field

Now that we have defined magnetization current, we find that Ampere’s Law for fields within some material becomes:

\[ \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \left( \mathbf{J}(\mathbf{r}) + \mathbf{J}_m(\mathbf{r}) \right) = \mu_0 \left( \mathbf{J}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r}) \right) \]

This of course is analogous to the expression we derived for Gauss’s Law in a dielectric media:

\[ \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_v(\mathbf{r}) + \rho_{vp}(\mathbf{r})}{\varepsilon_0} = \frac{\rho_v(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r})}{\varepsilon_0} \]

Recall that we removed the polarization charge from this expression by defining a new vector field \( \mathbf{D}(\mathbf{r}) \), leaving us with the more general expression of Gauss’s Law:

\[ \nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_v(\mathbf{r}) \]

**Q:** Can we similarly define a new vector field to "take care" of magnetization current??

**A:** Yes! We call this vector field the magnetic field \( \mathbf{H}(\mathbf{r}) \).
Let's begin by rewriting Ampere's Law as:

$$\nabla \times \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{J}_m(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

Yuck! Now we see clearly the problem. In free space, if we know current distribution $\mathbf{J}(\mathbf{r})$, we can find the resulting magnetic flux density $\mathbf{B}(\mathbf{r})$ using the Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \int \int \mathbf{J}(\mathbf{r}') \times \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

But this is the solution for current in free space! It is no longer valid if some material is present!

**Q:** Why?

**A:** Because, the magnetic flux density produced by current $\mathbf{J}(\mathbf{r})$ may magnetize the material (i.e., produce magnetic dipoles), thus producing magnetization currents $\mathbf{J}_m(\mathbf{r})$.

These magnetization currents $\mathbf{J}_m(\mathbf{r})$ will also produce a magnetic flux density—a modification of vector field $\mathbf{B}(\mathbf{r})$ that is not accounted for in the Biot-Savart expression shown above!

To determine the correct solution, we first recall that:

$$\mathbf{J}_m(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r})$$
Therefore Ampere’s Law is:

\[ \nabla \times B(\vec{r}) - \mu_0 \nabla \times M(\vec{r}) = \mu_0 J(\vec{r}) \]

\[ \nabla \times \left[ B(\vec{r}) - \mu_0 M(\vec{r}) \right] = \mu_0 J(\vec{r}) \]

\[ \nabla \times \left[ \frac{B(\vec{r})}{\mu_0} - M(\vec{r}) \right] = J(\vec{r}) \]

Now let’s define a new vector field \( \vec{H}(\vec{r}) \), called the magnetic field:

\[
\vec{H}(\vec{r}) = \frac{B(\vec{r})}{\mu_0} - M(\vec{r}) \quad \left[ \frac{\text{Amps}}{\text{meter}} \right]
\]

Ampere’s Law therefore can be written in terms of the magnetic field as:

\[ \nabla \times \vec{H}(\vec{r}) = J(\vec{r}) \]

Hey! We know what the solution to this differential equation is! Recall the solution to:

\[ \nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r}) \]

is the Biot–Savart Law.
If we make the substitution:

\[ H(\vec{r}) \leftrightarrow \frac{B(\vec{r})}{\mu_0} \]

we find that both differential equations are identical. Therefore their solutions are also identical when making the same substitution.

Making this substitution into the Biot-Sarvart Law, we find that:

\[
H(\vec{r}) = \frac{1}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'
\]

Q: Swell. But may I remind you that we were suppose to be finding the solution for the &%*@!+*#& magnetic flux density \( B(\vec{r}) \)!
True! But since we can find \( H(\vec{r}) \) from \( J(\vec{r}) \), our task now is to determine the relationship between \( B(\vec{r}) \) and \( H(\vec{r}) \).

We call the relationship between \( B(\vec{r}) \) and \( H(\vec{r}) \) a constitutive equation. For most media, we find that the magnetization vector \( M(\vec{r}) \) is directly proportional to the magnetic field \( H(\vec{r}) \):

\[
M(\vec{r}) = \chi_m \ H(\vec{r})
\]

where the proportionality coefficient \( \chi_m \) is the magnetic susceptibility of the material.

* Note that for a given magnetic field \( H(\vec{r}) \), as \( \chi_m \) increases, the magnetization vector \( M(\vec{r}) \) increases.

* Magnetic susceptibility \( \chi_m \) therefore indicates how susceptible the material is to magnetization.

* In other words, \( \chi_m \) is a measure of how easily (or difficult) it is to create and align magnetic dipoles (from atoms/molecules) within the material.

Again, note the analogy to electrostatics. We defined earlier electric susceptibility \( \chi_e \), which indicates how susceptible a material is to polarization (i.e., the creation of electric dipoles).

We can now determine the relationship between \( B(\vec{r}) \) and \( H(\vec{r}) \). Using the above expression, we find:
Hey! Magnetic field $\mathbf{H}(\vec{r})$ and magnetic flux density are related by a simple constant!

$$B(\vec{r}) = \mu H(\vec{r})$$

where:

$$\mu \doteq \text{material permeability} \quad \left[\frac{N}{A^2} = \frac{\text{Henries}}{m}\right]$$

$$= \mu_0 (1 + \chi_m)$$

We typically further simplify this expression by defining a relative permeability:
\[ \mu_r = \text{relative permeability} = 1 + \chi_m \]

So that:
\[ B(\vec{r}) = \mu H(\vec{r}) = \mu_0 \mu_r H(\vec{r}) \]

In other words, if the relative permeability of some material was, say, \( \mu_r = 2 \), then the permeability of the material is twice that of the permeability of free space (i.e., \( \mu = 2\mu_0 \)). This perhaps is more readily evident when we write:

\[ \mu_r = \frac{\mu}{\mu_0} \]

Note that \( \mu \) and/or \( \mu_r \) are proportional to magnetic susceptibility \( \chi_m \). As a result, permeability is likewise an indication of how susceptible a material to magnetization.

* If \( \mu_r = 1 \), this susceptibility is that of free space (i.e., none!).

* Alternatively, a large \( \mu_r \) indicates a material that is easily magnetized.

* For example, the relative permeability of iron is \( \mu_r = 4000 \)!
Now, we are finally able to determine the magnetic flux density in some material, produced by current density \( \mathbf{J}(\mathbf{r}) \)!

Since \( \mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r}) \) and:

\[
\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \iiint_{V} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'
\]

we find the desired solution:

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_{V} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'
\]

Comparing this result with the Biot-Sarvart Law for free space, we see that the only difference is that \( \mu_0 \) has been replaced with \( \mu \)!

This last result is therefore a more general form of the Biot-Savart Law, giving the correct result for fields within some material with permeability \( \mu \). Of course, the “material” could be free space. However, the expression above will still provide the correct answer; because for free space \( \mu = \mu_0 \), thus returning the equation to its original (i.e., free space) form!
Summarizing, we can attribute the existence of a magnetic field \( H(\vec{r}) \) to conduction current \( J(\vec{r}) \), while we attribute the existence of magnetic flux density to the total current density, including the magnetization current.

\[
J(\vec{r}) \Rightarrow H(\vec{r})
\]

\[
J(\vec{r}) + J_m(\vec{r}) \Rightarrow B(\vec{r})
\]

Finally, we again want to note the analogies between electrostatics and the magnetostatic expressions derived in this handout:

\[
B(\vec{r}) = \mu_0 H(\vec{r}) + \mu_0 M(\vec{r}) \quad \iff \quad D(\vec{r}) = \varepsilon_0 E(\vec{r}) + P(\vec{r})
\]

\[
B(\vec{r}) = \mu_0 (1 + \chi_m) H(\vec{r}) \quad \iff \quad D(\vec{r}) = \varepsilon_0 (1 + \chi_e) E(\vec{r})
\]

\[
B(\vec{r}) = \mu H(\vec{r}) \quad \iff \quad D(\vec{r}) = \varepsilon E(\vec{r})
\]

\[
B(\vec{r}) \iff D(\vec{r})
\]

\[
H(\vec{r}) \iff E(\vec{r})
\]

\[
M(\vec{r}) \iff P(\vec{r})
\]

\[
\chi_m \iff \chi_e
\]

\[
\mu \iff \varepsilon
\]
Example: Magnetization Currents

Problem:

Consider an infinite cylinder made of magnetic material. This cylinder is centered along the z-axis, has a radius of 2 m, and a permeability of $4\mu_0$.

Inside the cylinder there exists a magnetic flux density:

$$B(\bar{r}) = \frac{8\mu_0}{\rho} \hat{a}_\phi \quad (\rho \leq 1)$$

Determine the magnetization current $J_{sm}(\bar{r}_s)$ flowing on the surface of this cylinder, as well as the magnetization current $J_m(\bar{r})$ flowing within the volume of this cylinder.

Solution:

First, we note that we must know the magnetization vector $\mathbf{M}(\bar{r})$ in order to find the magnetization currents:

$$J_m(\bar{r}) = \nabla \times \mathbf{M}(\bar{r}) = \begin{bmatrix} A \\ m^2 \end{bmatrix}$$

$$J_{sm}(\bar{r}_s) = \mathbf{M}(\bar{r}_s) \times \hat{a}_n = \begin{bmatrix} A \\ m \end{bmatrix}$$
But, we must know the magnetic susceptibility \( \chi_m \) and the magnetic field \( H(\vec{r}) \) to determine magnetization vector.

\[
M(\vec{r}) = \chi_m \, H(\vec{r})
\]

Likewise, we need to know the relative permeability \( \mu_r \) to determine magnetic susceptibility:

\[
\chi_m = \mu_r - 1
\]

and we need to know the magnetic flux density \( B(\vec{r}) \) to determine the magnetic field:

\[
H(\vec{r}) = \frac{B(\vec{r})}{\mu}
\]

But guess what! We know the relative permeability \( \mu_r \) of the material, as well as the magnetic flux density within it!

\[
\mu = 4\mu_0, \quad \therefore \mu_r = 4
\]

\[
B(\vec{r}) = \frac{8\mu_0}{\rho} \hat{a}_\phi \quad (\rho \leq 1)
\]

Therefore, the magnetic field is:

\[
H(\vec{r}) = \frac{B(\vec{r})}{\mu} = \frac{1}{4\mu_0} \frac{8\mu_0}{\rho} \hat{a}_\phi = \frac{2}{\rho} \hat{a}_\phi
\]
and the magnetic susceptibility is:

\[ \chi_m = \mu_r - 1 = 4 - 1 = 3 \]

So the magnetization vector is:

\[ M(\vec{r}) = \chi_m H(\vec{r}) = (3) \frac{2}{\rho} \hat{a}_\phi = \frac{6}{\rho} \hat{a}_\phi \]

Now (finally!) we can determine the magnetization currents:

\[ J_m(\vec{r}) = \nabla \times M(\vec{r}) \]
\[ = \nabla \times \left( \frac{6}{\rho} \hat{a}_\phi \right) \]
\[ = 0 \]

The volume magnetization current density is zero—there is no magnetization current flowing within the cylinder!

Q: *No magnetization currents! So we’re done right? This problem is solved?*
**A:** Not hardly! Although there are no magnetization currents flowing within the cylinder, there might be magnetization currents flowing on the cylinder surface (i.e., \( J_{sm}(\vec{r}_s) \))!

\[
J_{sm}(\vec{r}_s) = \mathbf{M}(\vec{r}_s) \times \hat{a}_n
\]

Note for this problem, the unit vector normal to the surface of the cylinder is \( \hat{a}_n = \hat{a}_\rho \).

Likewise, the magnetization vector evaluated at the cylinder surface (i.e., at \( \rho = 2 \)) is:

\[
\mathbf{M}(\vec{r}_s) = \mathbf{M}(\rho = 2) = \frac{6}{\rho} \hat{a}_\phi \bigg|_{\rho=2} = 3 \hat{a}_\phi
\]

Therefore, the magnetization current density on the cylinder surface is:

\[
J_{sm}(\rho = 2) = \mathbf{M}(\rho = 2) \times \hat{a}_n = 3 \hat{a}_\phi \times \hat{a}_\rho = -3 \hat{a}_z \quad [A/m]
\]

Now, we're finally done.
Permanent Magnets

For **most** magnetic material (i.e., where $\mu \neq \mu_0$), we find that the magnetization vector $\mathbf{M}(\mathbf{r})$ will return to **zero** when a magnetization field $\mathbf{B}_m(\mathbf{r})$ is removed. In other words, the **magnetic dipoles** will vanish, or at least return to their random state.

For $\mathbf{M}(\mathbf{r}) \neq 0$, $\mathbf{B}_m(\mathbf{r}) \neq 0$

For $\mathbf{M}(\mathbf{r}) = 0$, $\mathbf{B}_m(\mathbf{r}) = 0$
However, some magnetic material, called ferromagnetic material, will retain its dipole orientation, even when the magnetizing field is removed!

In this case, a permanent magnet is formed (just like the ones you stick on your fridge)!

Ferromagnetic materials have numerous applications. For example, they will attract magnetic material.

Q: How?

A: A permanent magnet will of course produce everywhere a magnetic flux density \( \mathbf{B}(\vec{r}) \), which we can either attribute to the magnetic dipoles within the material, or to the equivalent magnetic current \( \mathbf{J}_m(\vec{r}) \).

The magnetic flux density produced by the magnet will act as a magnetizing field for some other magnetic material nearby, thus creating a second magnetization current \( \mathbf{J}_m(\vec{r}) \) within the nearby material. The magnetization currents of the material and the magnet will attract!
Another interesting application of ferromagnetic material is in non-volatile data storage (e.g., tape or disk). Ferromagnetics can be used as binary memory!
Q: *How?*

A: Recall that the magnetization vector in ferromagnetic material retains its direction after the magnetizing field $B_m(\vec{r})$ has been removed. In other words, it “remembers” the direction of the magnetizing field.

We can assign each of two different magnetizing directions, therefore, a binary state:

If ferromagnetic material is embedded in a tape or disk, we can magnetize (e.g., write) small sections of the media, or detect the magnetization (e.g., read) small sections of the media.

*Figure 8-21*
Field Equations in Magnetic Materials

Now that we have defined a magnetic field $H(\vec{r})$ and material permeability $\mu(\vec{r})$, we can write the magnetostatic (point form) equations for fields in magnetic material.

$$\nabla \times H(\vec{r}) = J(\vec{r})$$

$$\nabla \cdot B(\vec{r}) = 0$$

$$B(\vec{r}) = \mu(\vec{r})H(\vec{r})$$

We likewise can express these equations in integral form as:

$$\oint_{C} H(\vec{r}) \cdot d\ell = I_{enc}$$

$$\iiint_{S} B(\vec{r}) \cdot d\mathbf{s} = 0$$

$$B(\vec{r}) = \mu(\vec{r})H(\vec{r})$$
First, note the new form of Ampere's Law:

\[ \oint_{C} \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} = I_{\text{enc}} \]

Where \( I_{\text{enc}} \) is the conduction current only (i.e., it does not include magnetization current!).

Again, note the analogies to the new form of Gauss's Law we derived for electrostatics:

\[ \oiint_{S} \mathbf{D}(\mathbf{r}) \cdot d\mathbf{s} = Q_{\text{enc}} \]

where \( Q_{\text{enc}} \) is the free-charge enclosed by surface \( S \).

Perhaps the most important result of expressing magnetostatic fields in terms of material permeability \( \mu(\mathbf{r}) \) is that we do not have to rederive any of the results from Chapter 7!

In Chapter 7, the "material" we were concerned with was free space. The permeability of free space is by definition, \( \mu(\mathbf{r}) = \mu_{0} \).

If the material is not free space, then we simply change the results of Chapter 7 to reflect the correct value of permeability \( \mu(\mathbf{r}) \).

For example, we found that the Biot-Savart Law becomes:
\[
\mathbf{B}(\mathbf{r}) = \frac{\mu I}{4\pi} \oint \frac{d\ell' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}
\]

magnetic vector potential is:

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'
\]

or the magnetic flux produced by a infinite line current is:

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu I}{2\pi \rho} \hat{a}_\phi
\]
Magnetic Boundary Conditions

Consider the interface between two different materials with dissimilar permeabilities:

\[ H_1(\vec{r}), B_1(\vec{r}) \]

\[ \mu_1 \]

\[ H_2(\vec{r}), B_2(\vec{r}) \]

\[ \mu_2 \]

Say that a magnetic field and a magnetic flux density is present in both regions.

**Q:** How are the fields in dielectric region 1 (i.e., \( H_1(\vec{r}), B_1(\vec{r}) \)) related to the fields in region 2 (i.e., \( H_2(\vec{r}), B_2(\vec{r}) \))?

**A:** They must satisfy the magnetic boundary conditions!
First, let's write the fields at the interface in terms of their normal (e.g., $H_n(\vec{r})$) and tangential (e.g., $H_t(\vec{r})$) vector components:

\[ H_1(\vec{r}) = H_{1t}(\vec{r}) + H_{1n}(\vec{r}) \]
\[ H_2(\vec{r}) = H_{2t}(\vec{r}) + H_{2n}(\vec{r}) \]

Our first boundary condition states that the **tangential** component of the magnetic field is *continuous* across a boundary. In other words:

\[ H_{1t}(\vec{r}_b) = H_{2t}(\vec{r}_b) \]

where $\vec{r}_b$ denotes to any point along the interface (e.g., material boundary).
The **tangential** component of the magnetic field on one side of the material boundary is **equal** to the tangential component on the **other** side!

We can likewise consider the **magnetic flux densities** on the material interface in terms of their **normal** and **tangential** components:

The second magnetic boundary condition states that the **normal** vector component of the **magnetic flux density** is **continuous** across the material boundary. In other words:

\[
B_{1n}(\vec{r}_b) = B_{2n}(\vec{r}_b)
\]

where \(\vec{r}_b\) denotes any point along the interface (i.e., the material boundary).
Since $\mathbf{B}(\vec{r}) = \mu \mathbf{H}(\vec{r})$, these boundary conditions can likewise be expressed as:

\[
\begin{align*}
\mathbf{H}_{1t}(\vec{r}_b) &= \mathbf{H}_{2t}(\vec{r}_b) \\
\frac{\mathbf{B}_{1t}(\vec{r}_b)}{\mu_1} &= \frac{\mathbf{B}_{2t}(\vec{r}_b)}{\mu_2}
\end{align*}
\]

and as:

\[
\begin{align*}
\mathbf{B}_{1n}(\vec{r}_b) &= \mathbf{B}_{2n}(\vec{r}_b) \\
\mu_1 \mathbf{H}_{1n}(\vec{r}_b) &= \mu_2 \mathbf{H}_{2n}(\vec{r}_b)
\end{align*}
\]

Note again the perfect analogy to the boundary conditions of electrostatics!
Finally, recall that if a layer of free charge were lying at a dielectric boundary, the boundary condition for electric flux density was modified such that:

\[ \hat{a}_n \cdot [\mathbf{D}_1(\vec{r}_b) - \mathbf{D}_2(\vec{r}_b)] = \rho_s(\vec{r}_b) \]

\[ \mathbf{D}_{1n}(\vec{r}_b) - \mathbf{D}_{2n}(\vec{r}_b) = \rho_s(\vec{r}_b) \]

There is an analogous problem in magnetostatics, wherein a surface current is flowing at the interface of two magnetic materials:

In this case the tangential components of the magnetic field will not be continuous!
Instead, they are related by the boundary condition:

\[ \hat{a}_n \times (\mathbf{H}_1 (\vec{r}_b) - \mathbf{H}_2 (\vec{r}_b)) = \mathbf{J}_s (\vec{r}_b) \]

This expression means that:

1) \( \mathbf{H}_{1t} (\vec{r}_b) \) and \( \mathbf{H}_{2t} (\vec{r}_b) \) point in the same direction.

2) \( \mathbf{H}_{1t} (\vec{r}_b) \) and \( \mathbf{H}_{2t} (\vec{r}_b) \) are orthogonal to \( \mathbf{J}_s (\vec{r}_b) \).

3) The difference between \( |\mathbf{H}_{1t} (\vec{r}_b)| \) and \( |\mathbf{H}_{2t} (\vec{r}_b)| \) is \( |\mathbf{J}_s (\vec{r}_b)| \).

Recall that \( \mathbf{H}(\vec{r}) \) and \( \mathbf{J}_s (\vec{r}) \) have the same units—Amperes/meter!

Note for this case, the boundary condition for the magnetic flux density remains unchanged, i.e.:

\[ B_{1n} (\vec{r}_b) = B_{2n} (\vec{r}_b) \]

regardless of \( \mathbf{J}_s (\vec{r}_b) \).