8-4 Magnetostatic Boundary Value Problems

Reading Assignment: pp. 260-263

Q:

A:

We must solve differential equations, and apply boundary conditions to find a unique solution.

Good news! In electrical and computer engineering, the sources $\mathbf{J}(\vec{r})$ are typically known (unlike sources $\rho_v(\vec{r})$).

This process is best demonstrated with an example:

Example: Magnetostatic Boundary Conditions
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Consider two magnetic materials, separated by some boundary:

Throughout region 1, there is a constant magnetic field:

\[ \mathbf{H}_1(\mathbf{r}) = 3 \hat{a}_x + 5 \hat{a}_z \quad \text{A/m} \]

\[ \mu_1 = 2 \mu_0 \]

On the interface (i.e., boundary) between the two regions, there flows a surface current:

\[ \mathbf{J}_s(\mathbf{r}_b) = 2 \hat{a}_y \quad \text{A/m} \]

\[ \mu_2 = 3 \mu_0 \]

\[ \mathbf{H}_2(\mathbf{r}) = ?? \]

Throughout region 1, there is a constant magnetic field:

\[ \mathbf{H}_1(\mathbf{r}) = 3 \hat{a}_x + 5 \hat{a}_z \quad (z > 0) \]

On the interface (i.e., boundary) between the two regions, there flows a surface current:
\[ J_s (\vec{r}) = \begin{cases} 0 & z > 0 \\ 2 \hat{a}_y & z = 0 \\ 0 & z < 0 \end{cases} \quad [A/m] \]

**Q:** What is \( H_z (\vec{r}) \) and \( B_z (\vec{r}) \) in region 2??

**A:** Let's apply the boundary conditions and find out!

At the interface (i.e., \( z=0 \)), we can state that:

\[ H_z (\vec{r}_b) = H_z (z=0) = H_{2x} (z=0) \hat{a}_x + H_{2y} (z=0) \hat{a}_y + H_{2z} (z=0) \hat{a}_z \]

and:

\[ B_z (\vec{r}_b) = B_z (z=0) = B_{2x} (z=0) \hat{a}_x + B_{2y} (z=0) \hat{a}_y + B_{2z} (z=0) \hat{a}_z \]

Therefore, we need to find the scalar components \( H_{2x} (z=0) \), \( B_{2x} (z=0) \), etc.

First, we note that \( \hat{a}_z \) is normal to the interface, while \( \hat{a}_y \) and \( \hat{a}_z \) are tangential.
Thus, from boundary condition:

\[ \hat{a}_n \times (H_1 (r_b) - H_2 (r_b)) = J_s (r_b) \]

where we note that \( \hat{a}_n = \hat{a}_z \), we find:

\[ \hat{a}_z \times (H_1 (z = 0) - H_2 (z = 0)) = J_s (z = 0) \]

\[ \hat{a}_z \times [(3 - H_{2x}) \hat{a}_x + (0 - H_{2y}) \hat{a}_y + (5 - H_{2z}) \hat{a}_z] = 2 \hat{a}_y \]

Thus, we can ascertain:

\[ (3 - H_{2x}) \hat{a}_y + H_{2y} \hat{a}_x = 2 \hat{a}_y \]

\[ (3 - H_{2x}) \hat{a}_y \cdot \hat{a}_x + H_{2y} \hat{a}_x \cdot \hat{a}_x = 2 \hat{a}_y \cdot \hat{a}_x \]

\[ H_{2y} = 0 \]

and likewise:

\[ (3 - H_{2x}) \hat{a}_y + H_{2y} \hat{a}_x = 2 \hat{a}_y \]

\[ (3 - H_{2x}) \hat{a}_y \cdot \hat{a}_x + H_{2y} \hat{a}_x \cdot \hat{a}_y = 2 \hat{a}_y \cdot \hat{a}_y \]

\[ 3 - H_{2x} = 2 \]

\[ H_{2x} = 1 \]
Therefore:

\[ H_{2x}(z = 0) = 1 \quad \text{and} \quad H_{2y}(z = 0) = 0 \]

**Q:** But what about scalar component \( H_{2z}(z = 0) \)?

**A:** We can find it using our second boundary condition:

\[ \mu_1 H_{1n}(\vec{r}_b) = \mu_2 H_{2n}(\vec{r}_b) \]

From which we find:

\[ \mu_1 H_{1z}(z = 0) \hat{a}_z = \mu_2 H_{2z}(z = 0) \hat{a}_z \]

\[ 2\mu_0 5 \hat{a}_z = 3\mu_0 H_{2z}(z = 0) \hat{a}_z \]

And therefore:

\[ 2\mu_0 5 \hat{a}_z \cdot \hat{a}_z = 3\mu_0 H_{2z}(z = 0) \hat{a}_z \cdot \hat{a}_z \]

\[ 2\mu_0 5 = 3\mu_0 H_{2z}(z = 0) \]

\[ H_{2z}(z = 0) = \frac{10}{3} \]

Thus, we find that:

\[ H_z(z = 0) = H_{2x}(z = 0) \hat{a}_x + H_{2y}(z = 0) \hat{a}_y + H_{2z}(z = 0) \hat{a}_z \]

\[ = \hat{a}_x + \frac{10}{3} \hat{a}_z \]
And since:

\[ B_z (z = 0) = \mu_2 H_z (z = 0) \]

We find:

\[ B_z (z = 0) = 3\mu_0 \left( \hat{a}_x + \frac{10}{3} \hat{a}_z \right) = 3\mu_0 \hat{a}_x + 10\mu_0 \hat{a}_z \]

**Q:** But these are the values of the fields at the interface—what are the fields throughout region 2?

**A:** Note that there are no conduction currents within region 2. Thus, we find within region 2:

\[ \nabla \times H_2 (\vec{r}) = 0 \quad (z < 0) \]

Note that a constant magnetic field will satisfy the above equation. Moreover, the following constant magnetic field will likewise satisfy our boundary condition \( H_z (z = 0) \):

\[ H_2 (\vec{r}) = \hat{a}_x + \frac{10}{3} \hat{a}_z \] \( [A/m] \)

In other words, the value of the magnetic field at the boundary is likewise the value of the magnetic field everywhere throughout region 2 (\( H_2 (\vec{r}) \) is a constant vector field!). The magnetic flux density is therefore:

\[ B_z (\vec{r}) = \mu_2 H_2 (\vec{r}) = 3\mu_0 \hat{a}_x + 10\mu_0 \hat{a}_z \] \( [\text{W/m}^2] \)