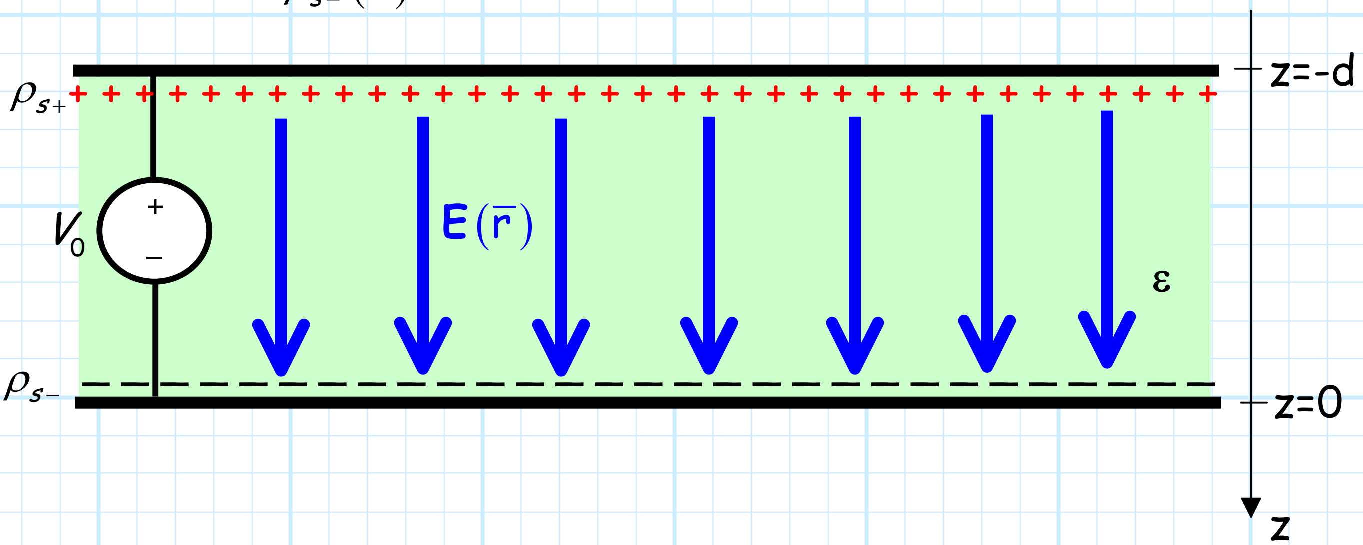
Energy Storage in Capacitors

Recall in a **parallel plate capacitor**, a surface charge distribution $\rho_{s+}(\bar{r})$ is created on **one** conductor, while charge distribution $\rho_{s-}(\bar{r})$ is created on the **other**.



Q: *How much **energy** is stored by these charges?*

A: Use the results of **Section 6-3**.

We learned that the energy **stored** by a **charge distribution** is:

$$W_e = \frac{1}{2}\iiint_V \rho_v(\bar{r})V(\bar{r})\ dv$$

The **equivalent** equation for **surface** charge distributions is:

$$W_e = \frac{1}{2} \iint_S \rho_s(\bar{r}) V(\bar{r})\, ds$$

For the parallel plate capacitor, we must integrate over **both** plates:

$$W_e = \frac{1}{2} \iint_{S_+} \rho_{s+}(\bar{r}) V(\bar{r})\, ds + \frac{1}{2} \iint_{S_-} \rho_{s-}(\bar{r}) V(\bar{r})\, ds$$

But on the **top** plate (i.e., S_+), we know that:

$$V(z\text{=-d}) = V_0$$

while on the **bottom** (i.e., S_-):

$$V(z\text{=}0) = 0$$

Therefore:

$$\begin{aligned} W_e &= \frac{V_0}{2} \iint_{S_+} \rho_{s+}(\bar{r})\, ds + \frac{0}{2} \iint_{S_-} \rho_{s-}(\bar{r})\, ds \\ &= \frac{V_0}{2} \iint_{S_+} \rho_{s+}(\bar{r})\, ds \end{aligned}$$

But, the remaining surface integral we know to be **charge** Q:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r})\, ds$$

Therefore, we find:

$$W_e = \frac{1}{2} V_0\, Q$$

But recall that:

$$Q = C\, V$$

where V is the **potential difference** between the two conductors (i.e., $V = V_0$).

Combining these **two** equations, we find:

$$\boxed{\begin{aligned} W_e &= \frac{1}{2} V_0 \, Q \\ &= \frac{1}{2} V_0 \left(C \, V \right) \\ &= \frac{1}{2} C \, V^2 \end{aligned}}$$

The above equation shows that the **energy stored** within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

Recall that we also can determine the stored energy from the **fields** within the dielectric:

$$W_e = \frac{1}{2} \iiint_V \mathbf{D}(\bar{\mathbf{r}}) \cdot \mathbf{E}(\bar{\mathbf{r}}) \, dv$$

Since the fields within the capacitor are **approximately**:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{V}{d} \hat{a}_z \qquad\qquad \mathbf{D}(\bar{\mathbf{r}}) = \frac{\varepsilon V}{d} \hat{a}_z$$

we find:

$$\begin{aligned} W_e &= \frac{1}{2}\iiint_V \mathbf{D}(\bar{\mathbf{r}})\cdot\mathbf{E}(\bar{\mathbf{r}})\,dv \\ &= \frac{1}{2}\iiint_V \frac{\varepsilon V^2}{d^2}\,dv \\ &= \frac{1}{2}\frac{\varepsilon V^2}{d^2}\iiint_V dv \\ &= \frac{1}{2}\frac{\varepsilon V^2}{d^2}(Volume) \end{aligned}$$

where the volume of the dielectric is simply the plate **surface area** S time the dielectric **thickness** d:

$$Volume = S\,d$$

Resulting in the expression:

$$\begin{aligned} W_e &= \frac{1}{2}\frac{\varepsilon V^2}{d^2}(S\,d) \\ &= \frac{1}{2}\frac{\varepsilon S}{d}V^2 \end{aligned}$$

Recall, however, that the **capacitance** of a parallel plate capacitor is:

$$C = \frac{\varepsilon S}{d}$$

Therefore:

$$W_e = \frac{1}{2}\frac{\varepsilon S}{d}V^2 = \frac{1}{2}CV^2$$

The **same** result as before!