Energy Storage in Capacitors

Recall in a parallel plate capacitor, a surface charge distribution $\rho_{s+}(\vec{r})$ is created on one conductor, while charge distribution $\rho_{s-}(\vec{r})$ is created on the other.

Q: How much energy is stored by these charges?

A: Use the results of Section 6-3.

We learned that the energy stored by a charge distribution is:

$$W_e = \frac{1}{2} \iiint_V \rho_{\nu}(\vec{r}) V(\vec{r}) \, dV$$

The equivalent equation for surface charge distributions is:
\[
W_e = \frac{1}{2} \iiint_{S} \rho_s(\vec{r}) V(\vec{r}) \, ds
\]

For the parallel plate capacitor, we must integrate over both plates:

\[
W_e = \frac{1}{2} \iiint_{S_+} \rho_{s+}(\vec{r}) V(\vec{r}) \, ds + \frac{1}{2} \iiint_{S_-} \rho_{s-}(\vec{r}) V(\vec{r}) \, ds
\]

But on the top plate (i.e., \(S_+\)), we know that:

\[
V(z=-d) = V_0
\]

while on the bottom (i.e., \(S_-\)):

\[
V(z=0) = 0
\]

Therefore:

\[
W_e = \frac{V_0}{2} \iiint_{S_+} \rho_{s+}(\vec{r}) \, ds + \frac{0}{2} \iiint_{S_-} \rho_{s-}(\vec{r}) \, ds
\]

\[
= \frac{V_0}{2} \iiint_{S_+} \rho_{s+}(\vec{r}) \, ds
\]

But, the remaining surface integral we know to be charge \(Q\):

\[
Q = \iiint_{S} \rho_{s+}(\vec{r}) \, ds
\]

Therefore, we find:

\[
W_e = \frac{1}{2} V_0 Q
\]

But recall that:

\[
Q = CV
\]
where $V$ is the **potential difference** between the two conductors (i.e., $V = V_0$).

Combining these **two** equations, we find:

\[
W_e = \frac{1}{2} V_0 Q
= \frac{1}{2} V_0 (C V)
= \frac{1}{2} C V^2
\]

The above equation shows that the **energy stored** within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

Recall that we also can determine the stored energy from the **fields** within the dielectric:

\[
W_e = \frac{1}{2} \iiint V \mathbf{D} (\mathbf{r}) \cdot \mathbf{E} (\mathbf{r}) \, dV
\]

Since the fields within the capacitor are **approximately**:

\[
\mathbf{E} (\mathbf{r}) = \frac{V}{d} \mathbf{\hat{a}_z} \quad \mathbf{D} (\mathbf{r}) = \frac{\varepsilon V}{d} \mathbf{\hat{a}_z}
\]

we find:
\[ W_e = \frac{1}{2} \iiint_V \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \, dv \]
\[ = \frac{1}{2} \iiint_V \varepsilon \mathbf{V}^2 \, dv \]
\[ = \frac{1}{2} \frac{\varepsilon \mathbf{V}^2}{d^2} \iiint_V dV \]
\[ = \frac{1}{2} \frac{\varepsilon \mathbf{V}^2}{d^2} (\text{Volume}) \]

where the volume of the dielectric is simply the plate surface area \( S \) time the dielectric thickness \( d \):

\[ \text{Volume} = S \, d \]

Resulting in the expression:

\[ W_e = \frac{1}{2} \frac{\varepsilon \mathbf{V}^2}{d^2} (S \, d) \]
\[ = \frac{1}{2} \frac{\varepsilon S}{d} \mathbf{V}^2 \]

Recall, however, that the capacitance of a parallel plate capacitor is:

\[ C = \frac{\varepsilon S}{d} \]

Therefore:

\[ W_e = \frac{1}{2} \frac{\varepsilon S}{d} \mathbf{V}^2 = \frac{1}{2} CV^2 \]

The same result as before!