The Stored Energy of Charge Distributions

Consider the case where a 1 Coulomb **point charge** is located at the **origin**. A **second** charge Q is **moved** to a distance r from the origin.

Q: *How much* ***energy*** *is stored in this simple charge distribution?*

A: Precisely the amount of **work** required to construct it !

Recall the amount of **work** required to move a charge Q through an electric field is:

$$W = -Q \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell}$$

The work required to **move** the a charge from **infinity** to a distance r from the origin is therefore:

$$W = -Q \int_{\infty}^{r} \mathbf{E}(\bar{\mathrm{r}}) \cdot \hat{a}_r \, dr$$

The 1 Coulomb charge at the origin of course produces the **electric field**:

$$\mathbf{E}(\bar{\mathrm{r}}) = \frac{\hat{a}_r}{4\pi\varepsilon r^2}$$

And produces the electric **potential** field:

$$V(\bar{\mathrm{r}}) = \frac{1}{4\pi\varepsilon\, r}$$

The work required to move charge Q to a distance r in **these** fields is:

$$\begin{aligned} W &= -Q\int_{\infty}^{r} \mathbf{E}(\bar{\mathrm{r}}) \cdot \hat{a}_r\, dr \\ &= -Q\int_{\infty}^{r} \frac{1}{4\pi\varepsilon r^2}\hat{a}_r \cdot \hat{a}_r\, dr \\ &= -\frac{Q}{4\pi\varepsilon}\int_{\infty}^{r} \frac{1}{r^2}\, dr \\ &= \frac{Q}{4\pi\varepsilon}\frac{1}{r} \end{aligned}$$

If we examine this result, we see that it is simply the **product** of the **charge** Q and the electric **potential** field $V(\bar{\mathrm{r}})$.

$$\begin{aligned} W &= \frac{Q}{4\pi\varepsilon}\frac{1}{r} \\ &= Q\,\frac{1}{4\pi\varepsilon\, r} \\ &= Q\, V(\bar{\mathrm{r}}) \end{aligned}$$

This seems to make sense! The units of electric potential are **Joules/Coulomb**, and the units of charge are of course **Coulombs**. The product of these two is therefore **energy**.

For a more general case, we find the **work** required to construct a **charge distribution** $\rho_v(\bar{r})$ is:

$$\boxed{W_e = \frac{1}{2}\iiint_V \rho_v(\bar{r})\, V(\bar{r})\; dv}$$

This equation, therefore, is **also** equal to the (potential) **energy stored** by this charge distribution!

$W_e =$ potential **energy** stored by a charge distribution

Recall that charge density is related to electric flux density via the point form of Gauss's Law:

$$\nabla \cdot \mathbf{D}(\bar{r}) = \rho_v(\bar{r})$$

Likewise, the electric field is related to the electric potential as:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$$

As shown on page 198, we can use these expressions to rewrite the stored energy in terms of the electric field and the electric flux density:

$$W_e = \frac{1}{2} \iiint_V \mathbf{D}(\bar{\mathbf{r}}) \cdot \mathbf{E}(\bar{\mathbf{r}}) \; dv$$

What these expressions mean is that it takes energy to assemble a charge distribution $\rho_v(\bar{\mathbf{r}})$, or equivalently, an electric field $\mathbf{E}(\bar{\mathbf{r}})$. This energy is stored until it is released—the charge density returns to zero.

Q: *Is this energy stored in the fields* $\mathbf{E}(\bar{\mathbf{r}})$ *and* $\mathbf{D}(\bar{\mathbf{r}})$*, or by the charge* $\rho_v(\bar{\mathbf{r}})$*??*

A: One equation for W_e would suggest that the energy is stored by the **fields**, while the other by the **charge**.

In turns out, **either** interpretation is correct! The fields $\mathbf{E}(\bar{\mathbf{r}})$ *and* $\mathbf{D}(\bar{\mathbf{r}})$ **cannot** exist without a charge density $\rho_v(\bar{\mathbf{r}})$, and knowledge of the fields allow us to determine **completely** the charge density.

In other words, charges and the fields they create are "inseparable pairs", since both must be present, we can attribute the stored energy to **either** quantity.