The Stored Energy of Charge Distributions

Consider the case where a 1 Coulomb point charge is located at the origin. A second charge $Q$ is moved to a distance $r$ from the origin.

**Q:** How much energy is stored in this simple charge distribution?

**A:** Precisely the amount of work required to construct it!

Recall the amount of work required to move a charge $Q$ through an electric field is:

$$ W = -Q \int_C \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} $$

The work required to move the a charge from infinity to a distance $r$ from the origin is therefore:

$$ W = -Q \int_{\infty}^{r} \mathbf{E}(\mathbf{r}) \cdot \mathbf{a}_r \, dr $$

The 1 Coulomb charge at the origin of course produces the electric field:

$$ \mathbf{E}(\mathbf{r}) = \frac{\hat{a}_r}{4\pi\varepsilon r^2} $$
And produces the electric **potential** field:

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon r} \]

The work required to move charge \( Q \) to a distance \( r \) in these fields is:

\[
W = -Q \int_\infty^r \mathbf{E}(\vec{r}) \cdot \hat{a}_r \, dr
\]

\[
= -Q \int_\infty^r \frac{1}{4\pi\varepsilon r^2} \hat{a}_r \cdot \hat{a}_r \, dr
\]

\[
= - \frac{Q}{4\pi\varepsilon} \int_\infty^r \frac{1}{r^2} \, dr
\]

\[
= \frac{Q}{4\pi\varepsilon} \frac{1}{r}
\]

If we examine this result, we see that it is simply the **product** of the **charge** \( Q \) and the electric **potential** field \( V(\vec{r}) \).

\[
W = \frac{Q}{4\pi\varepsilon} \frac{1}{r}
\]

\[
= Q \frac{1}{4\pi\varepsilon} \frac{1}{r}
\]

\[
= Q V(\vec{r})
\]

This seems to make sense! The units of electric potential are **Joules/Coulomb**, and the units of charge are of course **Coulombs**. The product of these two is therefore **energy**.
For a more general case, we find the work required to construct a charge distribution $\rho_v(\vec{r})$ is:

$$W_e = \frac{1}{2} \iiint_V \rho_v(\vec{r}) V(\vec{r}) \, dV$$

This equation, therefore, is also equal to the (potential) energy stored by this charge distribution!

\[ W_e = \text{potential energy stored by a charge distribution} \]

Recall that charge density is related to electric flux density via the point form of Gauss's Law:

$$\nabla \cdot \mathbf{D}(\vec{r}) = \rho_v(\vec{r})$$

Likewise, the electric field is related to the electric potential as:

$$\mathbf{E}(\vec{r}) = -\nabla V(\vec{r})$$

As shown on page 198, we can use these expressions to rewrite the stored energy in terms of the electric field and the electric flux density:
\[ W_e = \frac{1}{2} \iiint_V \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \, dV \]

What these expressions mean is that it takes energy to assemble a charge distribution \( \rho_v(\mathbf{r}) \), or equivalently, an electric field \( \mathbf{E}(\mathbf{r}) \). This energy is stored until it is released—the charge density returns to zero.

**Q:** Is this energy stored in the fields \( \mathbf{E}(\mathbf{r}) \) and \( \mathbf{D}(\mathbf{r}) \), or by the charge \( \rho_v(\mathbf{r}) \)?

**A:** One equation for \( W_e \) would suggest that the energy is stored by the fields, while the other by the charge.

In turns out, either interpretation is correct! The fields \( \mathbf{E}(\mathbf{r}) \) and \( \mathbf{D}(\mathbf{r}) \) cannot exist without a charge density \( \rho_v(\mathbf{r}) \), and knowledge of the fields allow us to determine completely the charge density.

In other words, charges and the fields they create are “inseparable pairs”, since both must be present, we can attribute the stored energy to either quantity.