2.1 The Inverting Configuration

Reading **Assignment**: *pp. 68-76*

One use of op-amps is to make **amplifiers**!

This seems rather obvious, but remember—an op-amp by itself has **too much gain** to be practical!

Thus, the op-amp is but **one element** in our amplifier design.

The resulting amplifier will be **very different** from the op-amp itself—do **not** confuse the op-amp with the amplifier!

In this section, we will consider the inverting amplifier—an amplifier constructed with **2 resistors** and **one op-amp**.

HO: ANALYSIS OF THE INVERTING AMPLIFIER

The inverting amplifier uses feedback—we **close a loop**!

HO: CLOSED-LOOP AND OPEN-LOOP GAIN

The result of this feedback is the **virtual short**.

HO: THE VIRTUAL SHORT

Let's determine the input and output **resistances** of the inverting amp!

HO: R_{IN} AND R_{OUT} OF THE INVERTING AMP

Make sure that your feedback is **negative**!

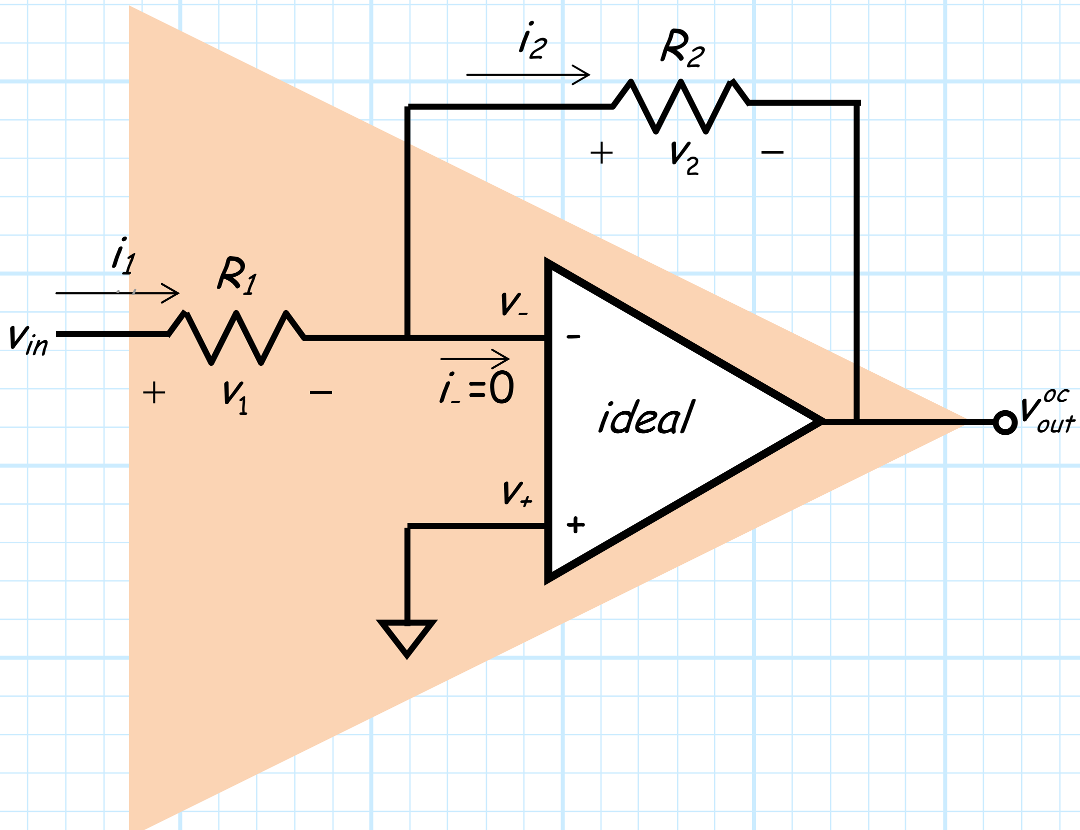
HO: FEEDBACK STABILITY

Another important application of the inverting configuration is the **weighted summer**.

HO: THE WEIGHTED SUMMER

Analysis of the Inverting Amplifier

Consider an **inverting amplifier**:



Note that we use here the **new notation** $v_+ = v_2$ and $v_- = v_1$.

Pay attention to your TA!

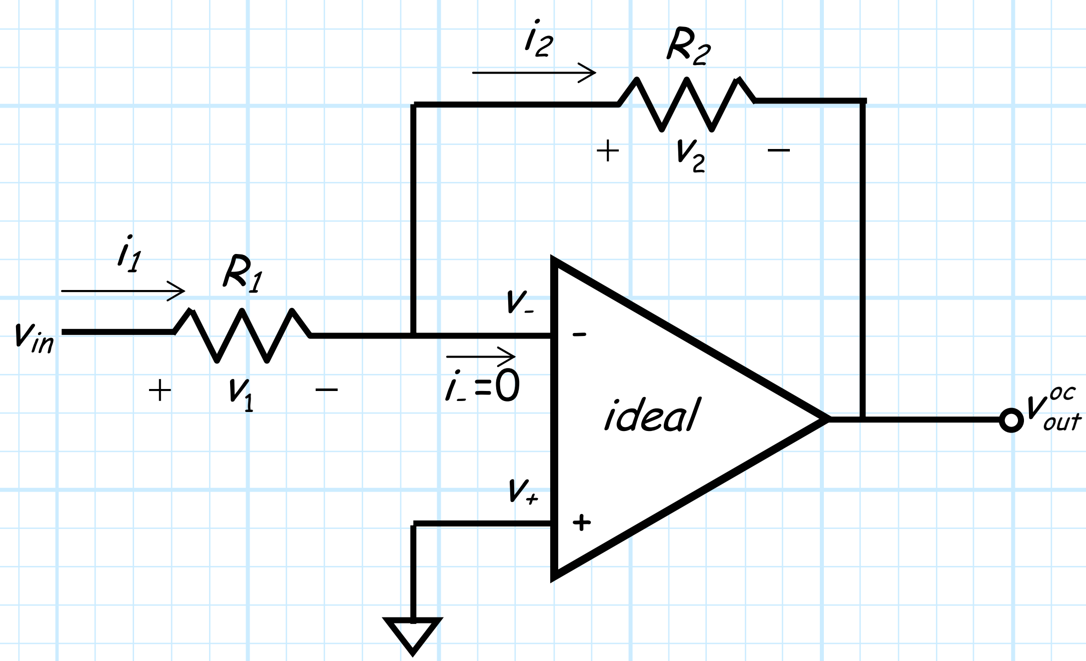
Now what is the **open-circuit voltage gain** of this inverting amplifier?

Let's start the analysis by writing down **all that we know**. First, the **op-amp equation**:

$$v_{out}^{oc} = A_{op}\left(v_+ - v_-\right)$$

Since the non-inverting terminal is **grounded** (i.e., $v_+=0$):

$$v_{out}^{oc} = -A_{op}\, v_-$$



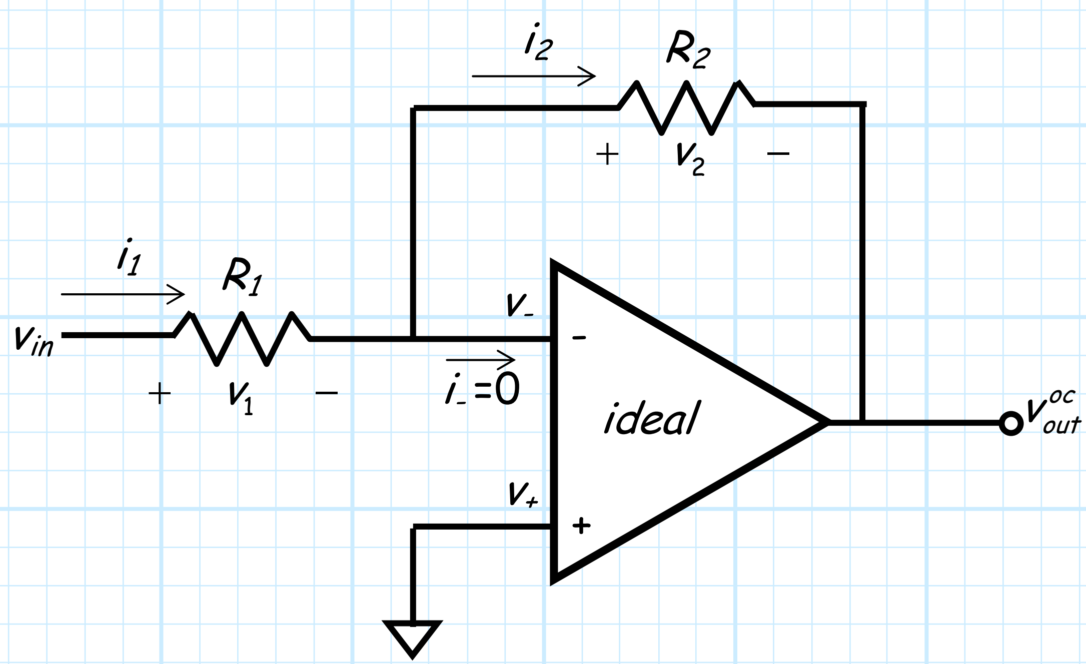
First some KCL…

Now let's apply our **circuit** knowledge to the remainder of the amplifier circuit. For example, we can use KCL to determine that:

$$i_1 = i_- + i_2$$

However, we know that the **input current** i_- of an ideal op-amp is **zero**, as the input resistance is infinitely large.

Thus, we reach the conclusion that: $i_1 = i_2$



And then some Ohm's law…

Likewise, we know from Ohm's Law:

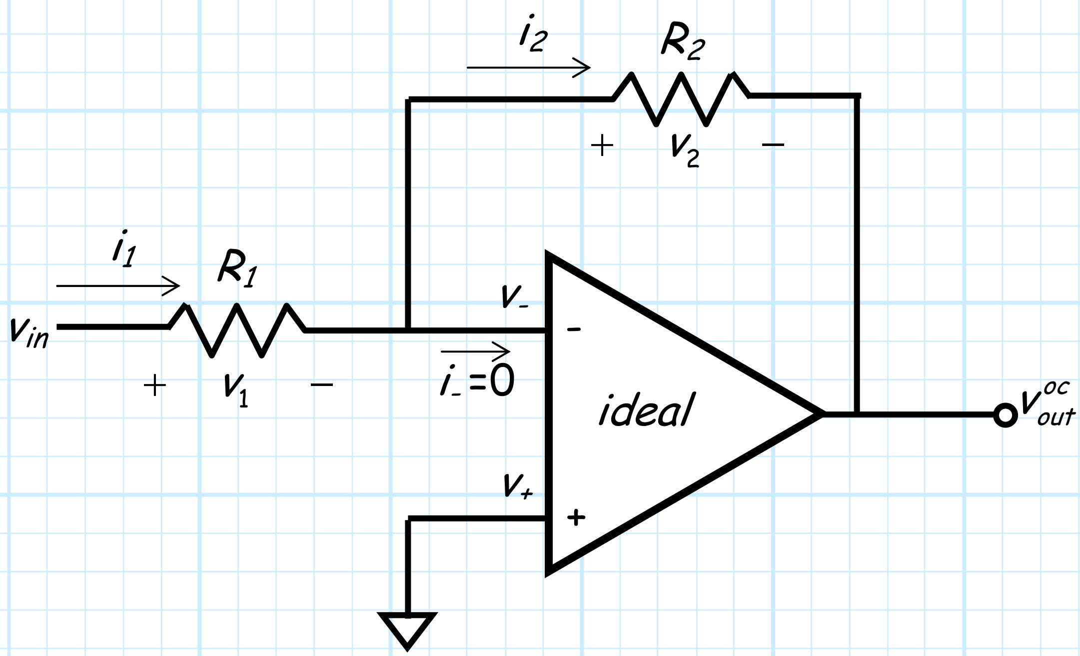
$$i_1 = \frac{v_1}{R_1}$$

and also that:

$$i_2 = \frac{v_2}{R_2}$$

And so combining:

$$\frac{v_1}{R_1} = \frac{v_2}{R_2}$$

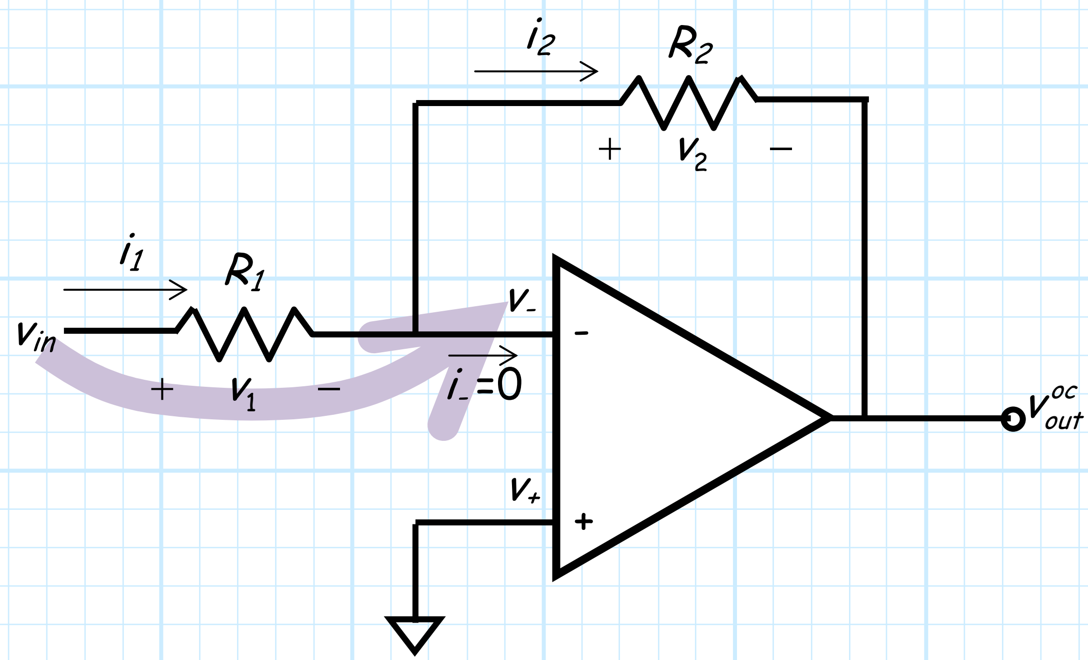


Followed by KVL…

Finally, from KCL we can conclude:

$$v_{in} - v_1 = v_- \qquad \Rightarrow \qquad v_1 = v_{in} - v_-$$

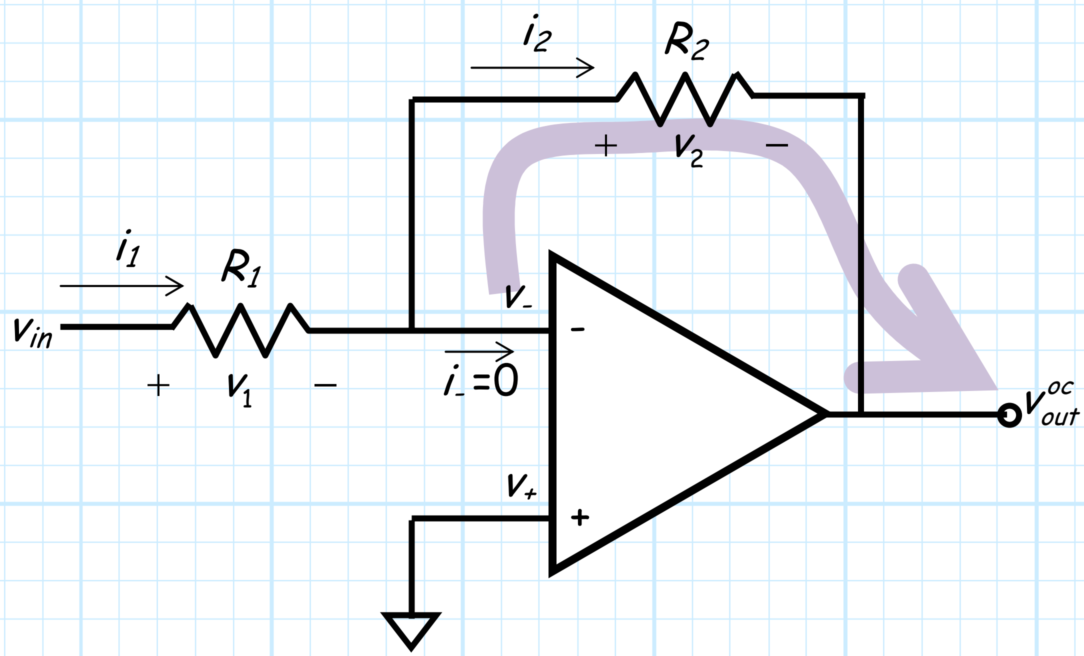
In other "words", we start at a potential of v_{in} volts (with respect to ground), we drop a potential of v_1 volts, and now we are at a potential of v_- volts (with respect to ground).



And yet another KVL…

Likewise, we start at a potential of of v_- volts (with respect to ground), we drop a potential of v_2 volts, and now we are at a potential of v_{out}^{oc} volts (with respect to ground).

$$v_- - v_2 = v_{out}^{oc} \qquad \Rightarrow \qquad v_2 = v_- - v_{out}^{oc}$$



The feed-back equation

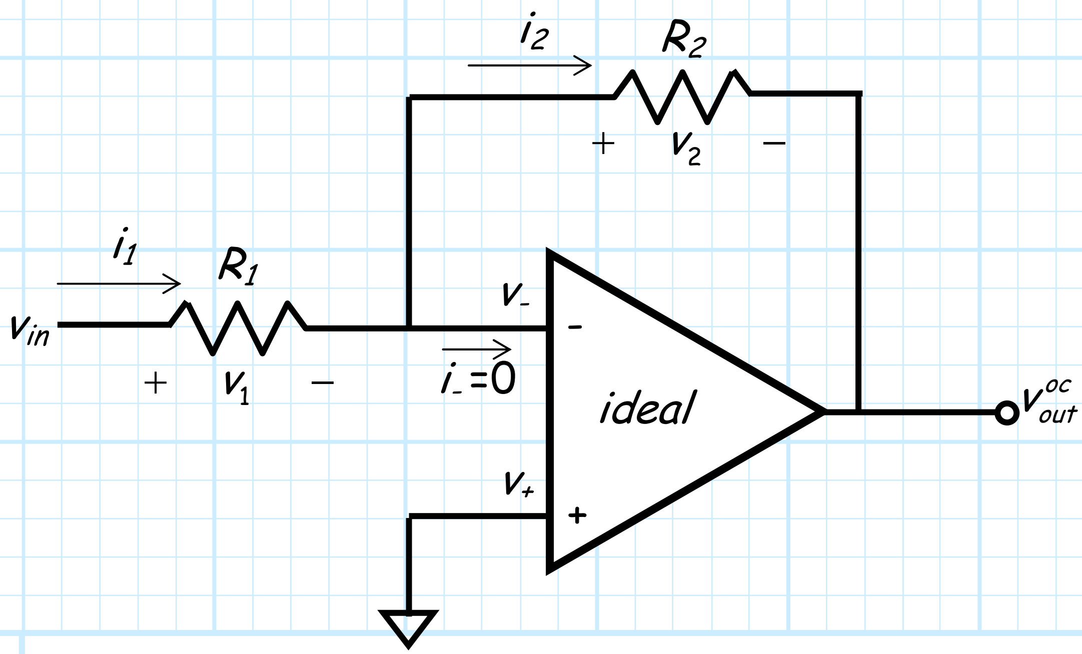
Combining these last three equations, we find:

$$\frac{v_{in} - v_-}{R_1} = \frac{v_- - v_{out}^{oc}}{R_2}$$

Now rearranging, we get what is known as the **feed-back equation**:

$$v_- = \frac{R_2 v_{in} + R_1 v_{out}^{oc}}{R_1 + R_2}$$

Note the feed-back equation relates v_- in terms of output v_{out}^{oc}.



The feed-forward equation

We can combine this feed-back equation with the **op-amp** equation:

$$v_{out}^{oc} = -v_{-} \, A_{op}$$

This op-amp equation is likewise referred to as the **feed-forward** equation.

Note this equation relates the output v_{out}^{oc} in terms of v_{-}.

We can combine the feed-**back** and feed-**forward** equations to determine an expression involving **only** input voltage v_{in} and output voltage v_{out}^{oc}:

$$\frac{R_2 \, v_{in} + R_1 \, v_{out}^{oc}}{R_1 + R_2} = -\frac{v_{out}^{oc}}{A_{op}}$$

...and the open-circuit voltage gain appears!

Rearranging this expression, we can determine the **output** voltage v_{out}^{oc} in terms of **input** voltage v_{in} :

$$v_{out}^{oc} = \left(\frac{-A_{op}\, R_2}{\left(R_1 + R_2\right) + A_{op}\, R_1} \right) v_{in}$$

and thus the **open-circuit voltage gain** of the inverting amplifier is:

$$A_{vo} = \frac{v_{out}^{oc}}{v_{in}} = \left(\frac{-A_{op}\, R_2}{\left(R_1 + R_2\right) + A_{op}\, R_1} \right)$$

Recall that the voltage gain A of an **ideal op-amp** is very large—approaching **infinity**.

Thus the open-circuit voltage gain of the **inverting amplifier** is:

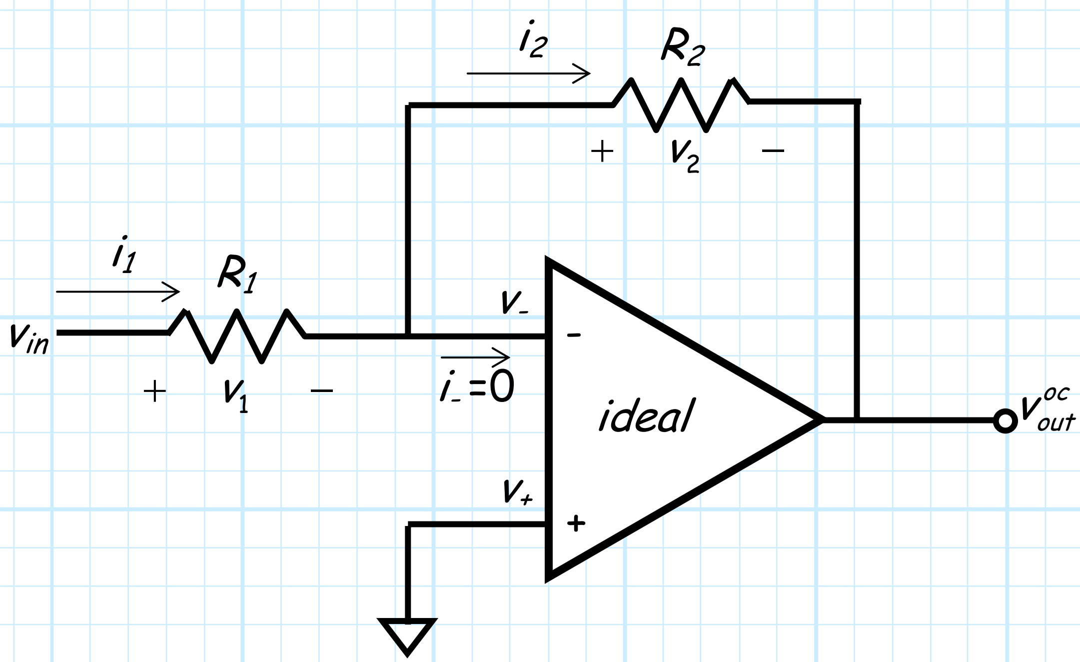
$$\begin{aligned} A_{vo} &= \lim_{A_{op} \to \infty} \left(\frac{-A_{op}\, R_2}{\left(R_1 + R_2\right) + A_{op} R_1} \right) \\ &= \frac{-R_2}{R_1} \end{aligned}$$

Summarizing

Summarizing, we find that for the **inverting amplifier**:

$$A_{vo} = \frac{-R_2}{R_1}$$

$$v_{out}^{oc} = \left(\frac{-R_2}{R_1}\right) v_{in}$$



The non-inverting terminal is at ground potential

One **last thing**. Let's use this final result to determine the value of v_-, the voltage at the **inverting** terminal of the op-amp.

Recall:

$$v_- = \frac{R_2\, v_{in} + R_1\, v_{out}^{oc}}{R_1 + R_2}$$

Inserting the equation:

$$v_{out}^{oc} = \left(\frac{-R_2}{R_1}\right) v_{in}$$

we find:

$$\begin{aligned} v_- &= \frac{R_2\, v_i + R_1 \left({-R_2}/{R_1} \right) v_{in}}{R_1 + R_2} \\ &= \frac{R_2\, v_{in} - R_2\, v_{in}}{R_1 + R_2} \\ &= 0 \end{aligned}$$

The voltage at the inverting terminal of the op-amp is **zero**!

The logic behind the virtual short

Thus, since the non-inverting terminal is grounded ($v_2 = 0$), we find that:

$$v_- = v_+ \qquad \text{and} \quad \therefore \qquad v_+ - v_- = 0$$

Recall that this should **not** surprise us.

We know that if **op-amp** gain A_{op} is infinitely large, its output v_{out}^{oc} will also be infinitely large (can you say saturation?), **unless** $v_+ - v_-$ is **infinitely small**.

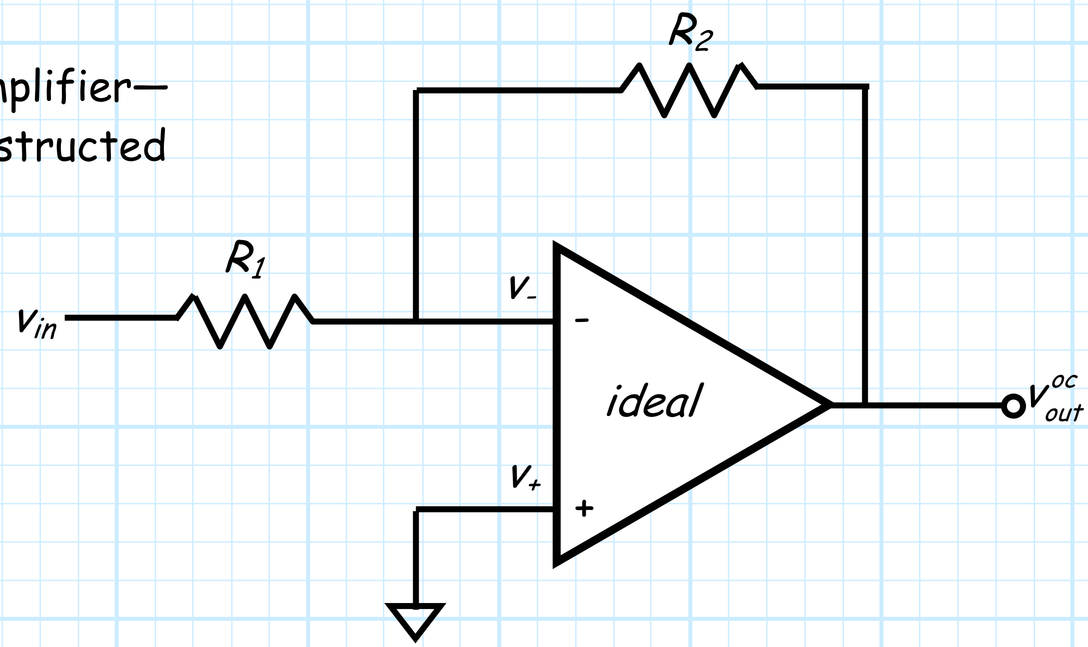
We find that the **actual** value of $v_+ - v_-$ to be:

$$v_+ - v_- = \frac{v_{out}^{oc}}{A_{op}} = \frac{-R_2}{A_{op} R_1} v_{in}$$

a number which approaches **zero** as $A_{op} \rightarrow \infty$!

Closed-Loop and Open-Loop Gain

Consider the inverting amplifier—
a **feedback** amplifier constructed
with an op-amp:



The **open-circuit** voltage gain of this amplifier:

$$A_{vo} = \frac{-R_2}{R_1}$$

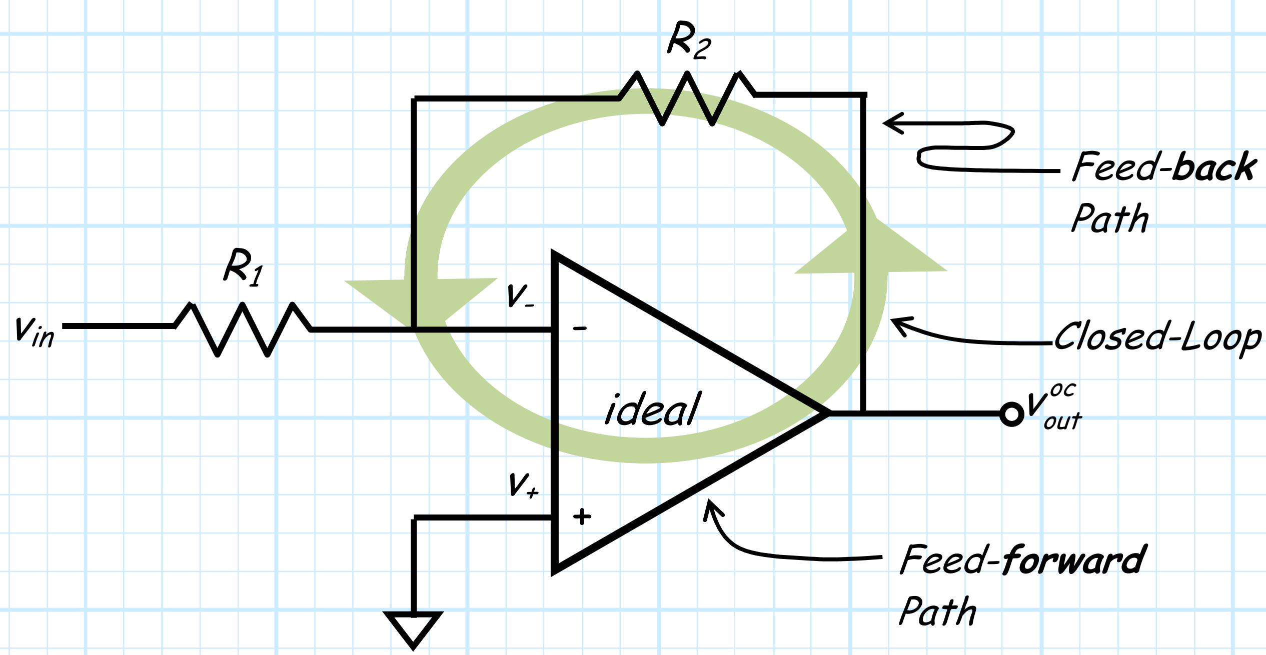
is also referred to by engineers the **closed loop gain** of the **feedback amplifier**.

A closed loop

Q: ***Closed loop**? What does that mean?*

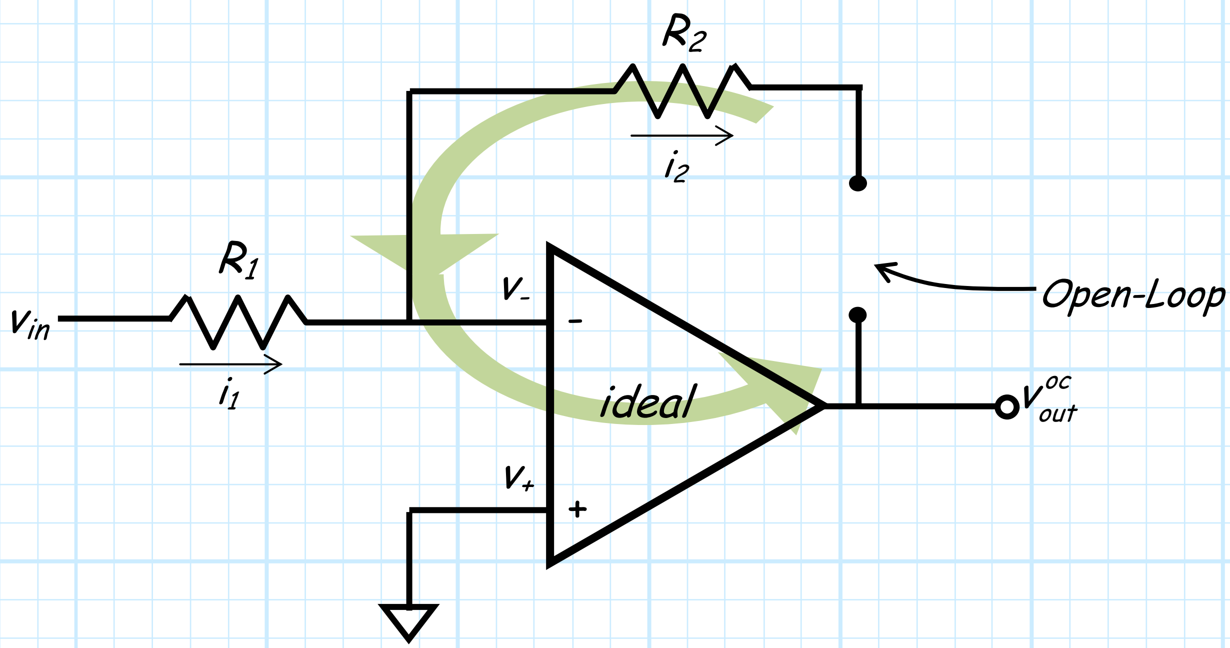
A: The term "closed loop" refers to loop formed by the **feed-forward** path and the **feed-back** (i.e., feedback) path of the amplifier.

In this case, the **feed-forward** path is formed by the **op-amp**, while the **feed-back** path is formed by the feedback **resistor** R_2.



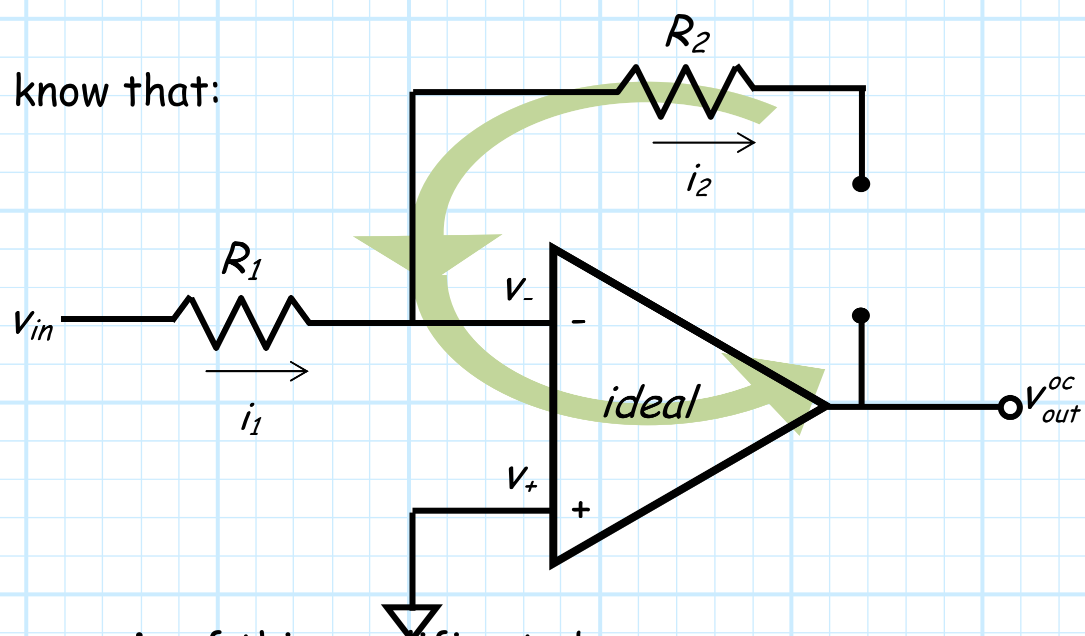
An open loop

If the loop is **broken**, then we say the loop is "open". The gain (v_o/v_i) for the open loop case is referred to as the **open-loop gain**.



Open and closed loop gains

For example, in the circuit we know that:



$$v_+ = 0$$
$$v_{out}^{oc} = A_{op}\left(v_+ - v_-\right)$$
$$i_1 = i_2 = 0$$
$$v_- = v_{in} - i_1 R_1 = 0$$

Combining, we find the **open-loop gain** of this amplifier to be:

$$A_{open} = \frac{v_{out}^{oc}}{v_{in}} = -A_{op}$$

Once we "close" the loop, we have an amplifier with a **closed-loop** gain:

$$A_{closed} = \frac{v_{out}^{oc}}{v_{in}} = -\frac{R_2}{R_1}$$

which of course is the **open-circuit voltage gain** of this inverting amplifier.

Feedback is a wonderful thing

Note that the **closed**-loop gain ($-R_2/R_1$) does **not** explicitly involve the op-amp gain A_{op}.

* The closed-loop gain is determined by two **resistor** values, which typically are selected to provide **significant** gain ($|A_{vo}| > 1$), albeit not so large that the amplifier is easily **saturated**.

* Conversely, the **open**-loop gain ($-A_{op}$) obviously **does** involve the op-amp gain. Moreover, as in this case, the open-loop gain of a feedback amplifier often **only** involves the op-amp gain!

* As a result, the **op-amp gain** is often alternatively referred to as the **open-loop gain**.

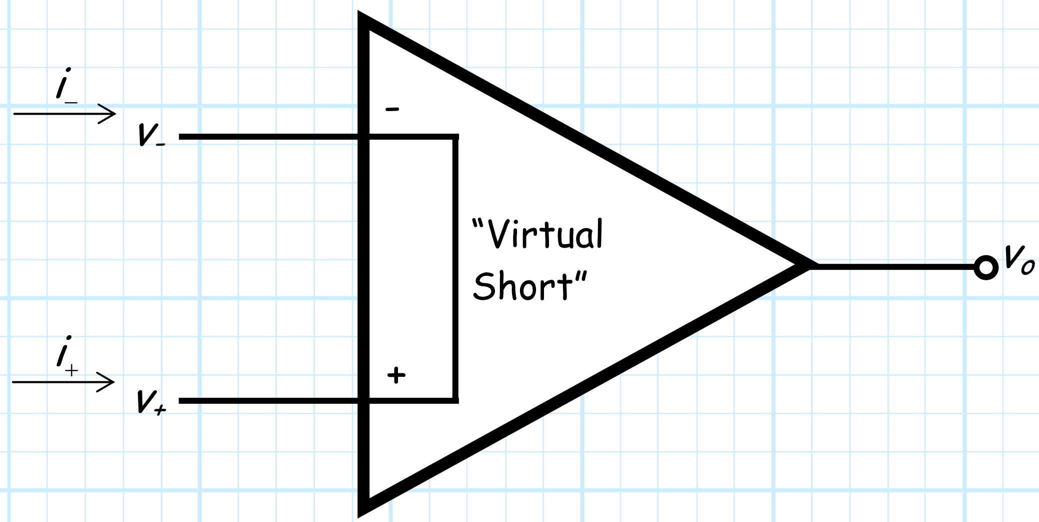
Note that **closing** the feedback loop turns a generally **useless** amplifier (the gain is too high!) into a **very** useful one (the gain is just right)!

The Virtual Short

For **feedback** amplifiers constructed with op-amps, we have found (and will continue to find) that the two op-amp terminals will always be **approximately equal** ($v_- \approx v_+$).

Of course, this must be true in order to avoid saturation, as the gain of an op-amp is ideally **infinitely** large.

Since $v_- \approx v_+$ for feedback applications, it **appears** that the two op-amp terminals are **shorted** together!

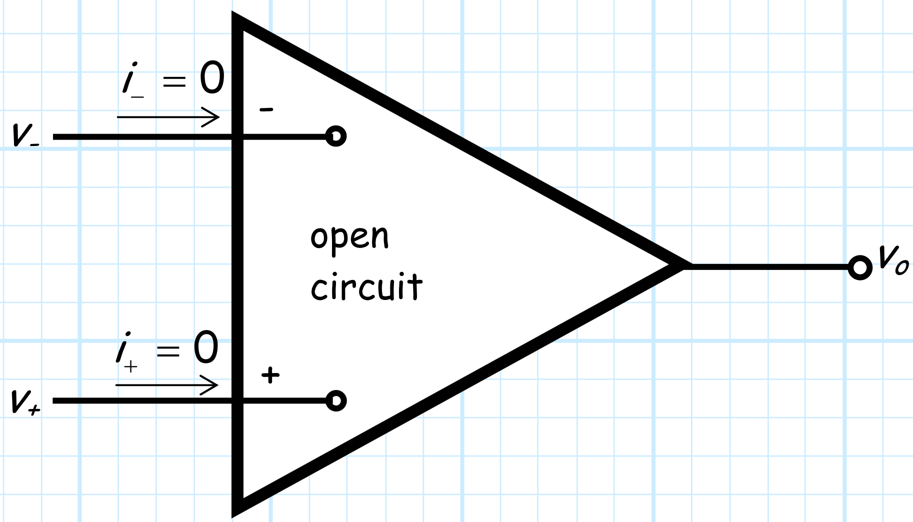


There is no short inside the op-amp!

The condition in op-amp feedback amplifiers where $v_1 = v_2$ is known as the "**virtual short**".

Remember, although the two input terminal **appear** to be shorted together, they are most certainly **not**!

If a **true** short **were** present, then **current** could flow from one terminal to the other (i.e., $i_- = -i_+$). However, we know that the input resistance of an op-amp is ideally **infinite**, and thus we know that the input current into an op-amp is **zero**.

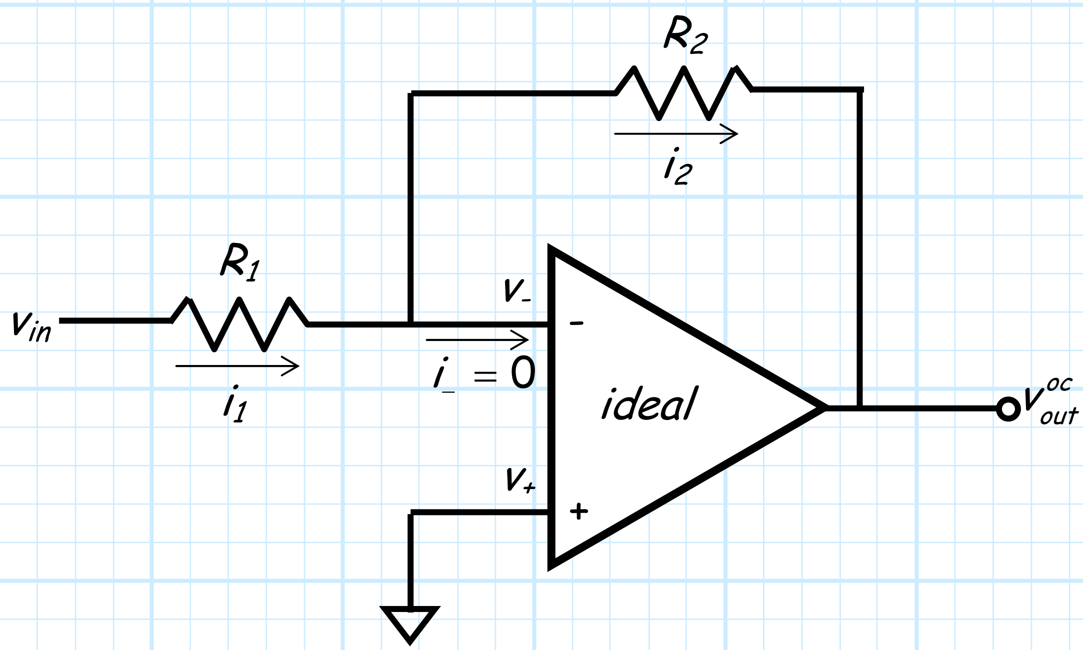


Therefore, it is **not** the op-amp that enforces the condition $v_1 = v_2$, it is the **feedback** that makes this so!

The virtual short: your new BFF

Applying the concept of a **virtual short** can greatly **simplify** the analysis of an op-amp feedback amplifier.

For example, consider **again** the **inverting amplifier**:



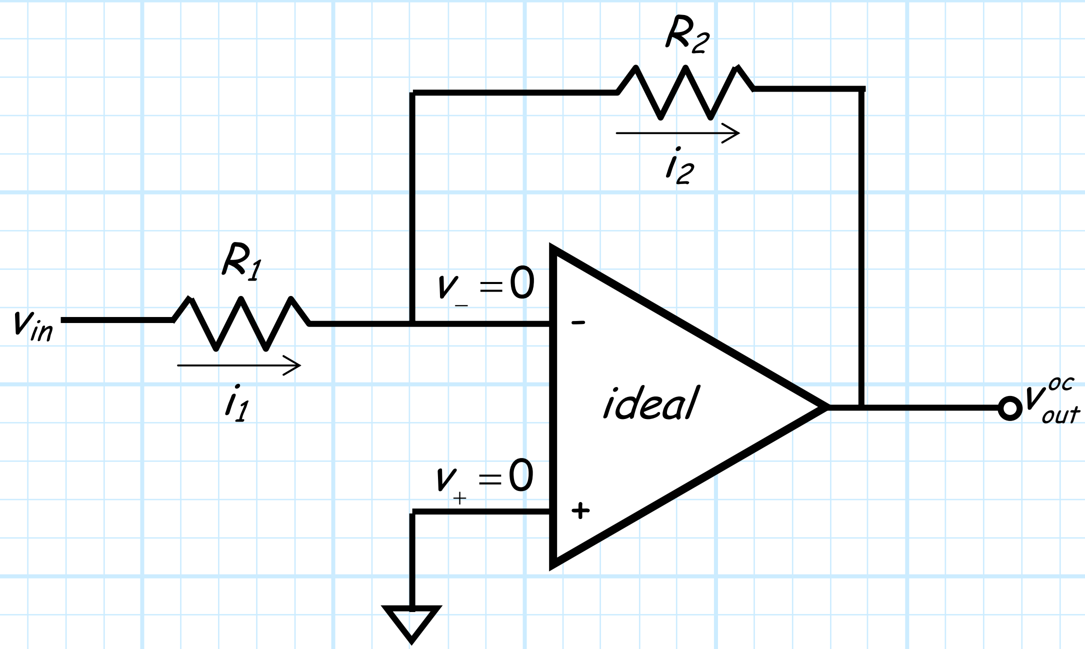
The virtual ground

This time, we **begin** the analysis by applying the **virtual short** condition:

$$v_- \cong v_+$$

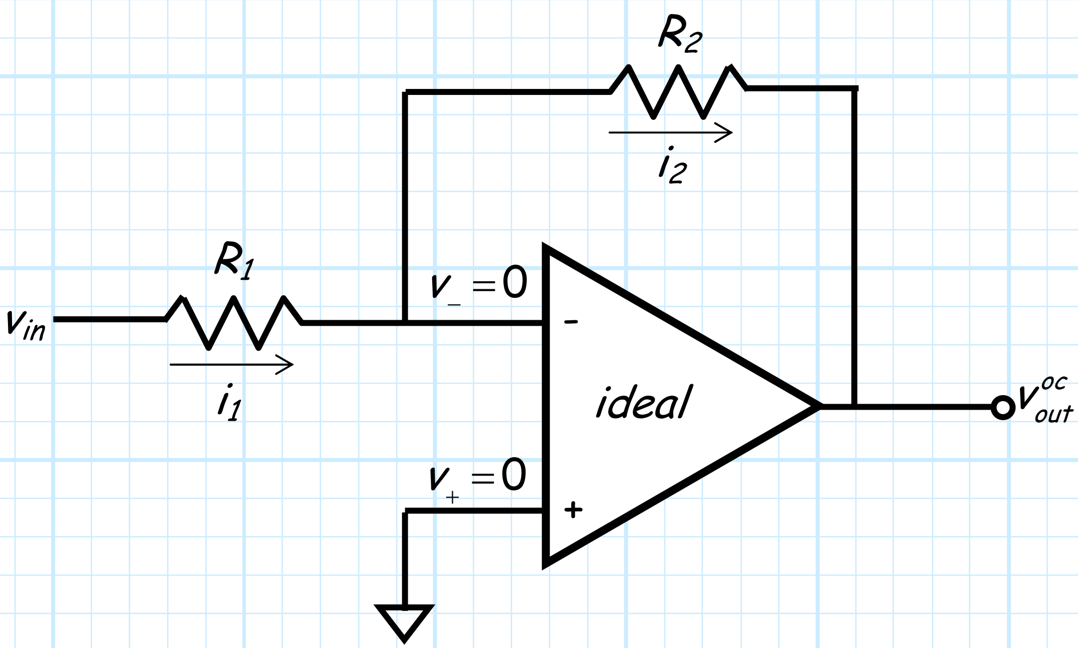
Since the non-inverting terminal is **grounded** ($v_+ = 0$), the virtual short means that the non-inverting terminal is **likewise** at zero potential ($v_- = 0$)!

We refer to this condition as a **virtual ground**.



Isn't this simpler?

Analyzing the remainder of the **circuit**, we find:



$$i_- = 0$$

$$i_1 = i_2 + i_- = i_2$$

$$i_1 = \frac{v_{in} - v_-}{R_1} = \frac{v_{in}}{R_1}$$

$$i_2 = \frac{v_- - v_{out}^{oc}}{R_2} = \frac{-v_{out}^{oc}}{R_2}$$

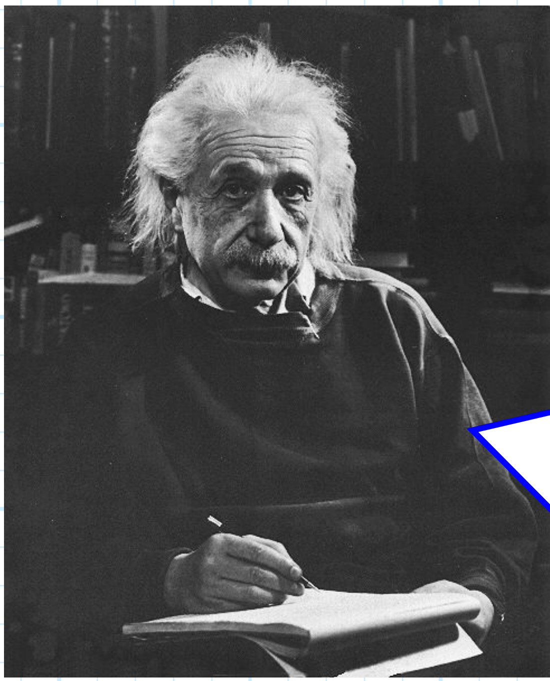
Combining, we find:

$$\frac{v_{in}}{R_1} = \frac{-v_{out}^{oc}}{R_2}$$

Rearranging, we again find the **open-circuit**, **closed-loop** voltage gain:

$$A_{vo} = \frac{v_{out}^{oc}}{v_{in}} = \frac{-R_2}{R_1}$$

Your TA is even smarter than this guy!



*Note this is **exactly** the result we found before, yet in this case we **never** considered the op-amp equation:*

$$v_{out}^{oc} = A_{op}\left(v_+ - v_-\right)$$

*The virtual short equation ($v_+ = v_-$) **replaced** the op-amp equation in our analysis of this **feedback amplifier**.*

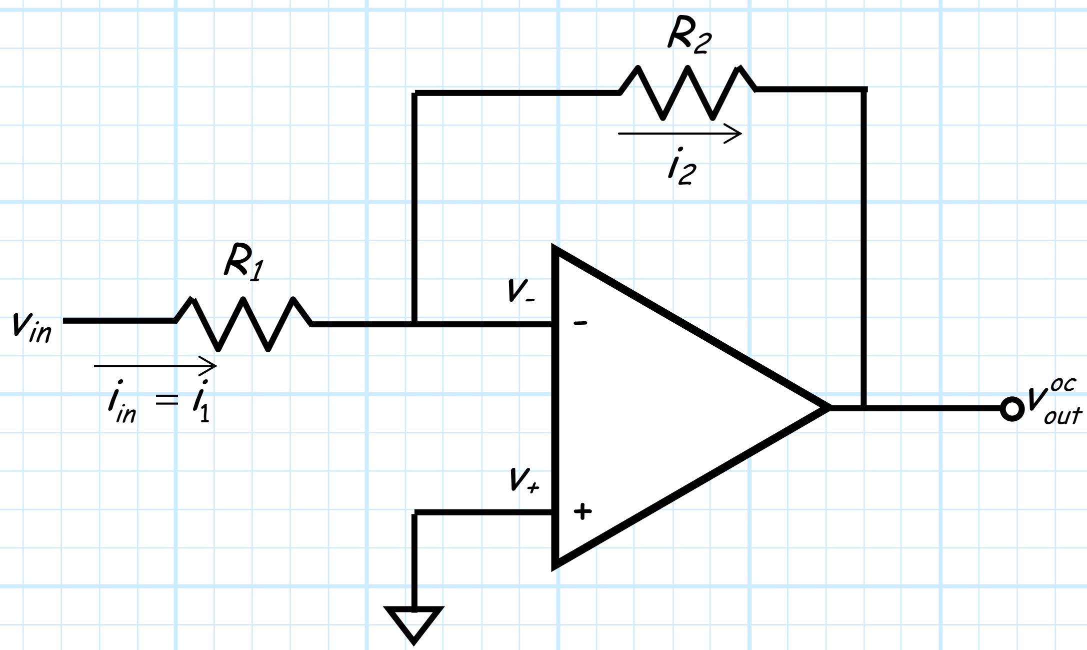
*Eeffectively, the two equations say the **same** thing (provided A_{op} is infinitely **large**, and we have **negative feedback** in the circuit).*

R_{in} and R_{out} of the Inverting Amplifier

Recall that the input resistance of an amplifier is:

$$R_{in} = \frac{v_{in}}{i_{in}}$$

For the **inverting** amplifier, it is evident that the input current i_{in} is equal to i_1:



Its input resistance

From **Ohm's Law**, we know that this current is:

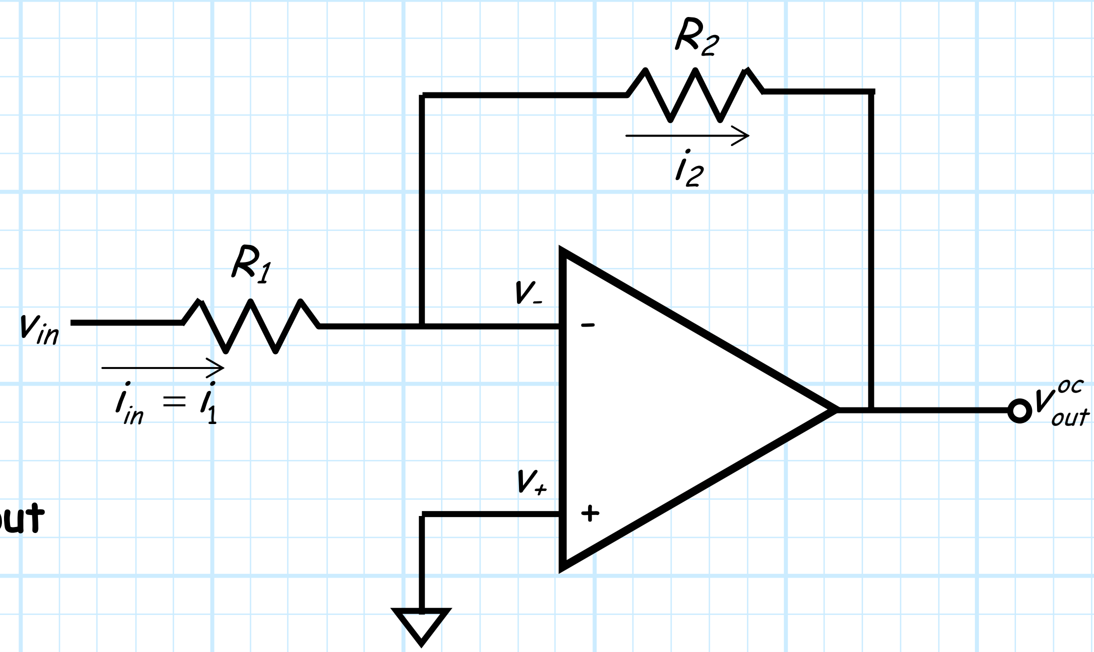
$$i_{in} = i_1 = \frac{v_{in} - v_1}{R_1}$$

The non-inverting terminal is "connected" to **virtual ground**:

$$v_- = 0$$

and thus the **input current** is:

$$i_{in} = i_1 = \frac{v_{in}}{R_1}$$



We now can determine the **input** resistance:

$$R_{in} = \frac{v_{in}}{i_{in}} = v_{in}\left(\frac{R_1}{v_{in}}\right) = R_1$$

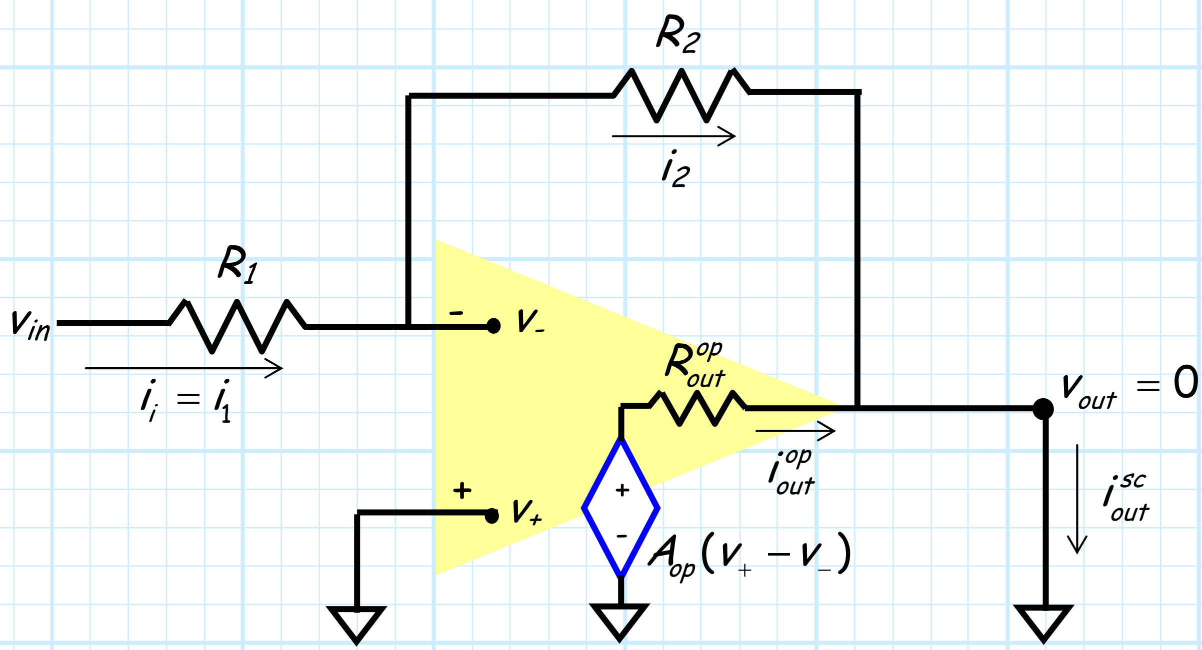
The **input resistance** of this inverting amplifier is therefore $R_{in} = R_1$!

Output resistance is harder

Now, let's attempt to determine the **output resistance** R_{out}.

Recall that we need to determine **two** values: the **short-circuit output current** $\left(i_{out}^{sc}\right)$ and the **open-circuit output voltage** $\left(v_{out}^{oc}\right)$.

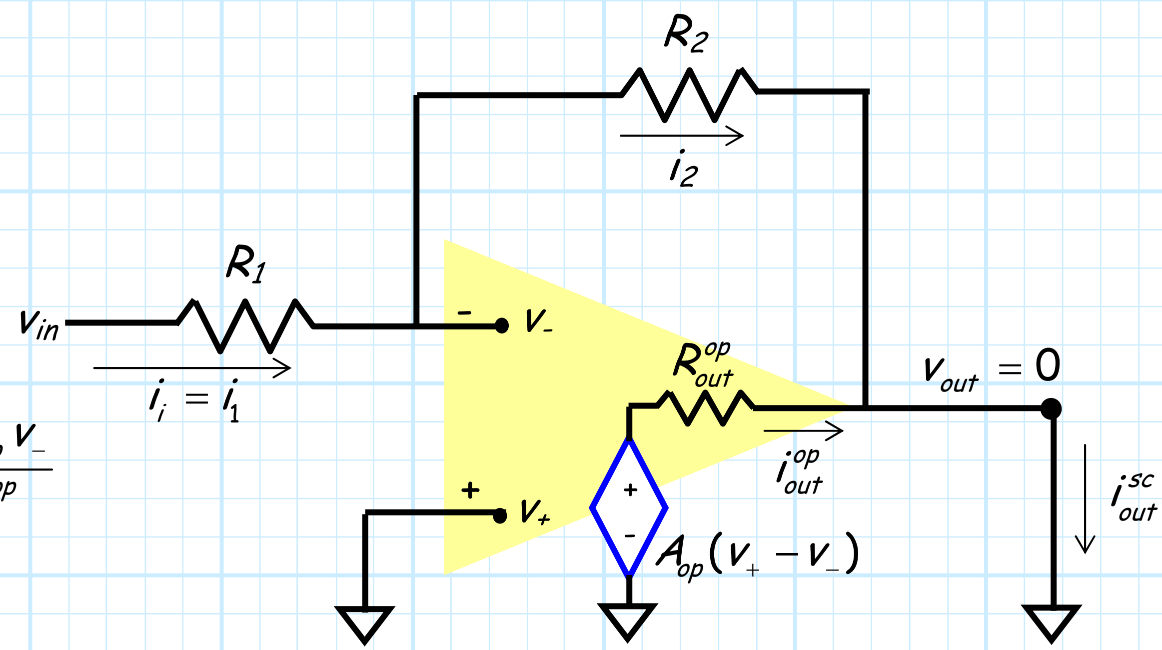
To accomplish this, we must replace the op-amp in the circuit with its **linear circuit model**:



First, the short circuit output current

From KCL, we find that:

$$i_{out}^{sc} = i_2 + i_{out}^{op}$$



where:

$$i_{out}^{op} = \frac{-A_{op}\, v_- - v_{out}^{oc}}{R_o^{op}} = \frac{-A_{op}\, v_-}{R_o^{op}}$$

and:

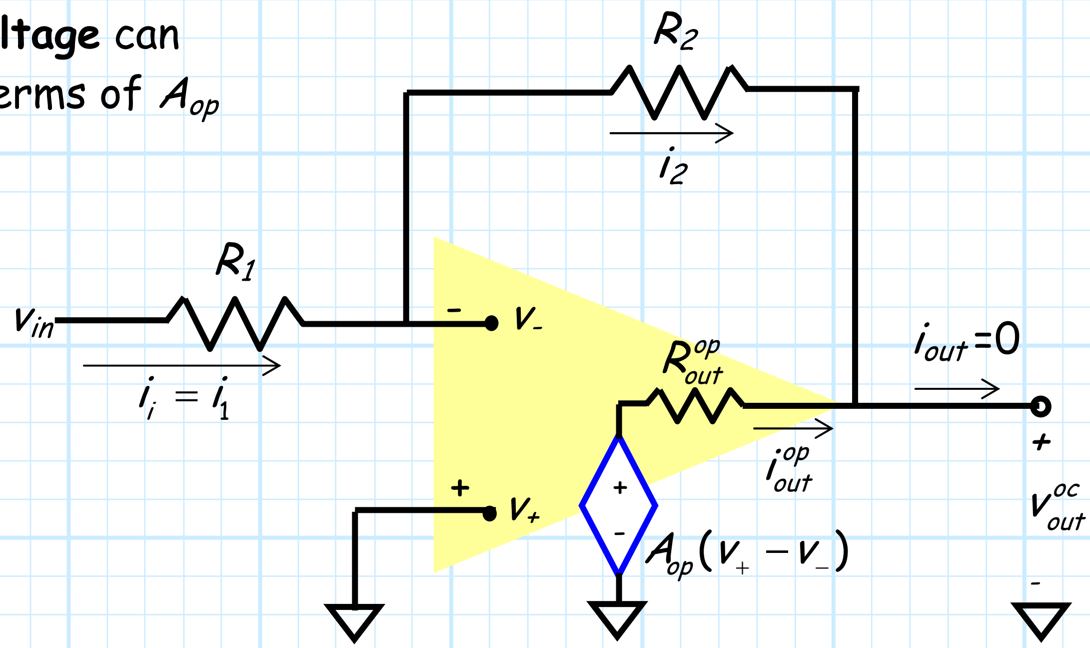
$$i_2 = \frac{v_- - v_{out}^{oc}}{R_2} = \frac{v_-}{R_2}$$

Therefore, the **short-circuit output current** is:

$$i_{out}^{sc} = \frac{v_-}{R_2} - \frac{A_{op}\, v_-}{R_{out}^{op}} = \left(\frac{R_{out}^{op} - R_2\, A_{op}}{R_2\, R_{out}^{op}}\right) v_- \cong -\frac{A_{op}}{R_{out}^{op}}\, v_-$$

Now, the open circuit output voltage

The **open-circuit output voltage** can likewise be determined in terms of A_{op} and v_- .



Here, it is evident that since $i_{out} = 0$:

$$i_2 = -i_{out}^{op}$$

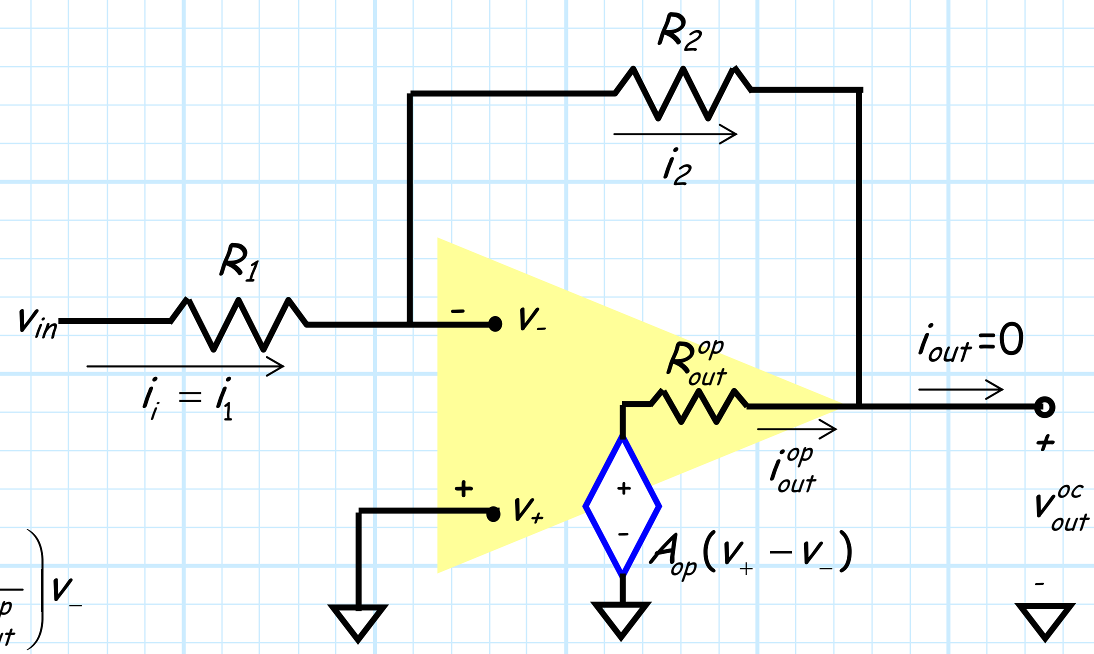
where we find from Ohm's Law:

$$i_2 = \frac{v_- - (-A_{op} v_-)}{R_2 + R_{out}^{op}} = \left(\frac{1 + A_{op}}{R_2 + R_{out}^{op}} \right) v_-$$

The open-circuit output voltage

Now from KVL:

$$v_{out}^{oc} = v_- - R_2\, i_2$$



Inserting the expression for i_2:

$$\begin{aligned} v_{out}^{oc} &= v_- - R_2\left(\frac{1+A_{op}}{R_2+R_{out}^{op}}\right)v_- \\ &= \left(\frac{R_2+R_{out}^{op}}{R_2+R_{out}^{op}} - \frac{R_2\left(1+A_{op}\right)}{R_2+R_{out}^{op}}\right)v_- \\ &= \left(\frac{R_o^{op} - R_2\, A_{op}}{R_2+R_{out}^{op}}\right)v_- \\ &\cong -\frac{R_2\, A_{op}}{R_2+R_{out}^{op}}\, v_- \end{aligned}$$

Now we find the output resistance

Now, we can find the **output resistance** of this amplifier:

$$
\begin{aligned}
R_{out} &= \frac{v_{out}^{oc}}{i_{out}^{sc}} \\
&= \left(\frac{-R_2\, A_{op}}{R_2 + R_o^{op}}\right)\left(\frac{-A_{op}}{R_o^{op}}\right)^{-1} \\
&= \frac{R_2\, R_o^{op}}{R_2 + R_o^{op}} \\
&= R_2 \,\|\, R_o^{op}
\end{aligned}
$$

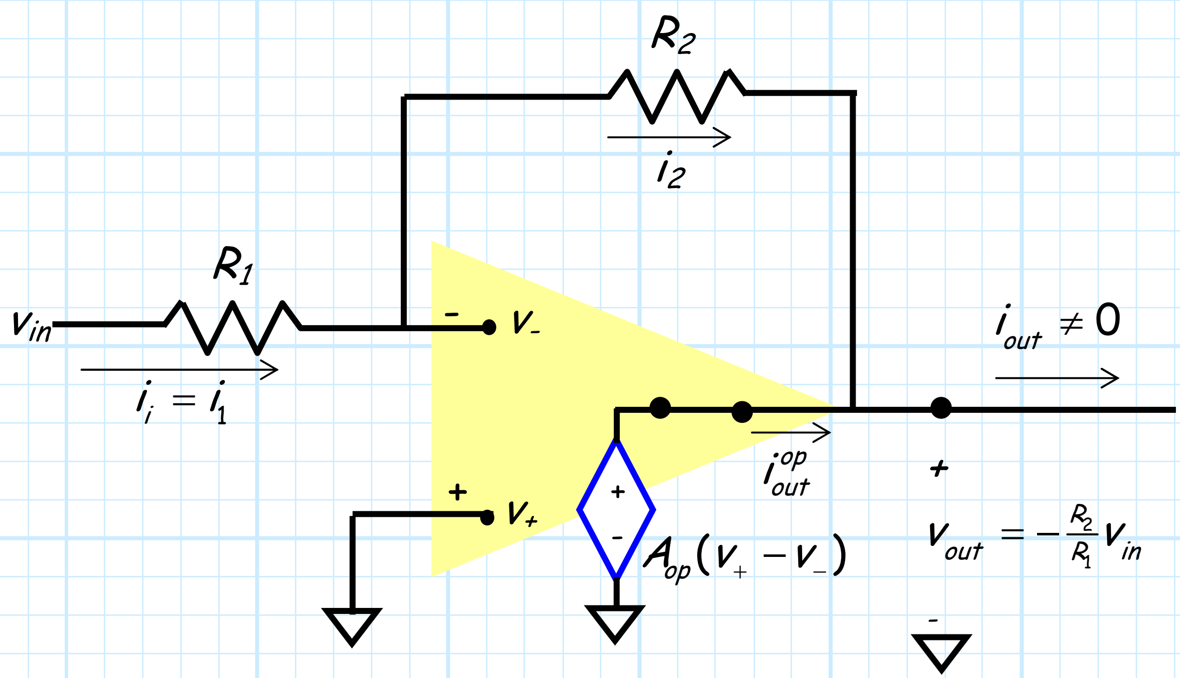
In other words, the **inverting amplifier output resistance** is simply equal to the value of the **feedback** resistor R_2 in **parallel** with **op-amp** output resistance R_{out}^{op}.

This is zero if the op-amp is ideal

Ideally, of course, the op-amp output resistance is **zero**, so that the output resistance of the inverting amplifier is **likewise zero**:

$$\begin{aligned} R_{out} &= R_2 \| R_{out}^{op} \\ &= R_2 \| 0 \\ &= 0 \end{aligned}$$

Note for this case—where the output resistance is **zero**—the output voltage will be the **same**, regardless of what **load** is attached at the output (e.g., **regardless** of i_{out})!



For real op-amps the output resistance is small

Thus, if $R_{out} = 0$, then the output voltage is equal to the **open-circuit** output voltage—even when the output is **not** open circuited:

$$v_{out} = -\frac{R_2}{R_1} v_{in} \qquad \text{for } \textbf{all} \quad i_{out} \quad !!$$

Recall that it is this property that made $R_{out} = 0$ an "ideal" amplifier characteristic.

We will find that real (i.e., non-ideal!) op-amps typically have an output resistance that is **very small** ($R_{out}^{op} \ll R_2$), so that the **inverting amplifier** output resistance is **approximately equal** to the op-amp output resistance:

$$\begin{aligned} R_{out} &= R_2 \| R_{out}^{op} \\ &\approx R_{out}^{op} \end{aligned}$$

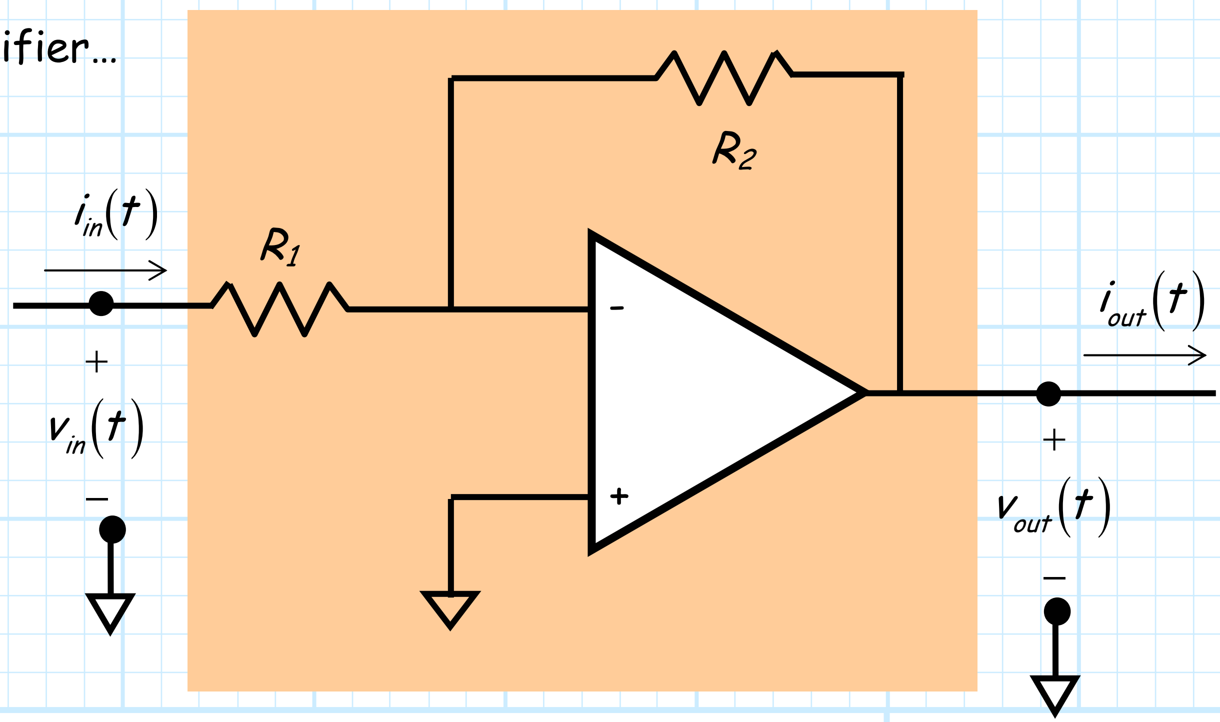
A summary

Summarizing, we have found that for the inverting amplifier:

$$R_{in} = R_1$$

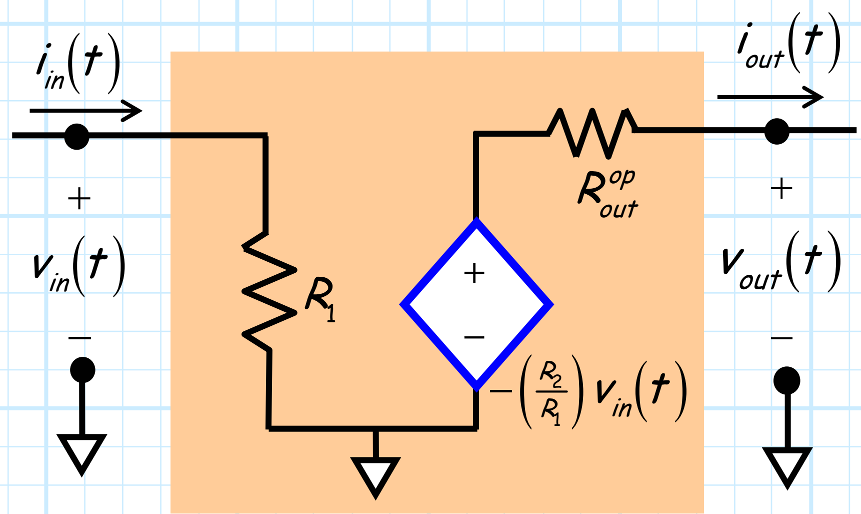
$$R_{out} \approx R_{out}^{op} \qquad \text{(ideally zero)}$$

Thus, **this** inverting amplifier…



The inverting amp equivalent circuit

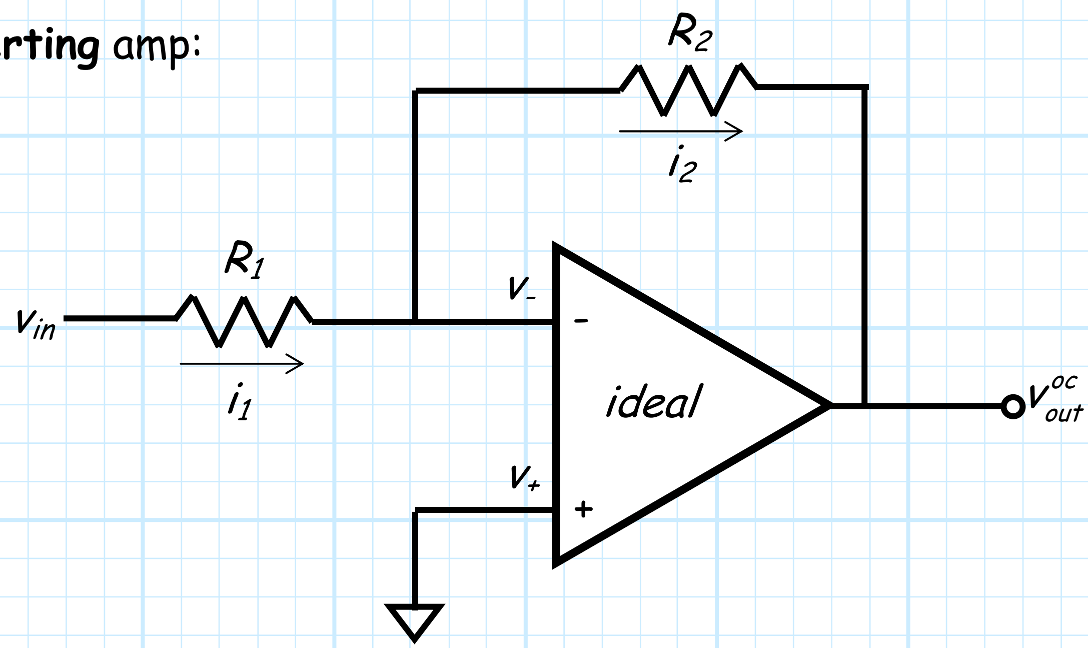
…has the **equivalent circuit**:



Note the input resistance and open-circuit voltage gain of the **inverting amplifier** is **VERY different** from that of the **op-amp** itself!

Feedback Stability

Recall that for the **inverting** amp:



we have the **feed-forward** equation:

$$v_{out}^{oc} = -A_{op}\, v_-$$

and the **feed-back** equation:

$$v_- = \frac{R_2\, v_{in} + R_1\, v_{out}^{oc}}{R_1 + R_2}$$

Nothing more fun than calculus!

Taking **derivatives** of these equations, we find that:

$$\frac{\partial v_{out}^{oc}}{\partial v_-} = \frac{\partial\left(-A_{op}\, v_-\right)}{\partial v_-} = -A_{op}$$

and:

$$\frac{\partial v_-}{\partial v_{out}^{oc}} = \frac{\partial}{\partial v_{out}^{oc}}\left(\frac{R_2\, v_{in} + R_1\, v_{out}^{oc}}{R_1 + R_2}\right) = \frac{R_1}{R_1 + R_2}$$

These derivatives are **very** important in determining the **stability** of the feedback amplifier.

Feed-forward

To see this, consider what happens when, for some reason, v_- **changes** some small value Δv_- from its nominal value of $v_-=0$.

The **output** voltage will then likewise change by a value Δv_{out}^{oc}:

$$\Delta v_{out}^{oc} \approx \left(\frac{\partial v_{out}^{oc}}{\partial v_-}\right)\Delta v_- = -A_{op}\,\Delta v_1$$

Note if Δv_- is **positive**, then Δv_{out}^{oc} will be **negative**—an **increase** in v_- leads to a **decrease** in v_{out}^{oc}.

This describes the feed-**forward** portion of the "loop."

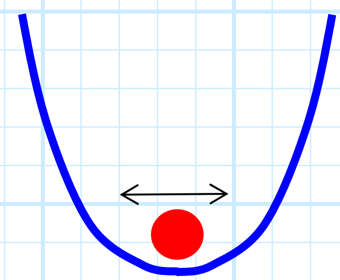
Feed-back

The feed-**back** equation states that a small change in the **output** voltage (i.e., Δv_{out}^{oc}), will likewise result in a small change in v_-:

$$\Delta v_- \approx \left(\frac{\partial v_-}{\partial v_{out}^{oc}}\right) \Delta v_{out}^{oc} = \left(\frac{R_1}{R_1 + R_2}\right) \Delta v_{out}^{oc}$$

Note in this case, a **decreasing** output voltage will result in a **decreasing** inverting terminal voltage v_-.

Thus, if the inverting terminal voltage tries to **increase** from its correct value of v_1=0, the control loop will react by **decreasing** the voltage v_1—essentially **counteracting** the initial change!



Negative feedback-in this case it's a good thing!

Note that the **loop product**:

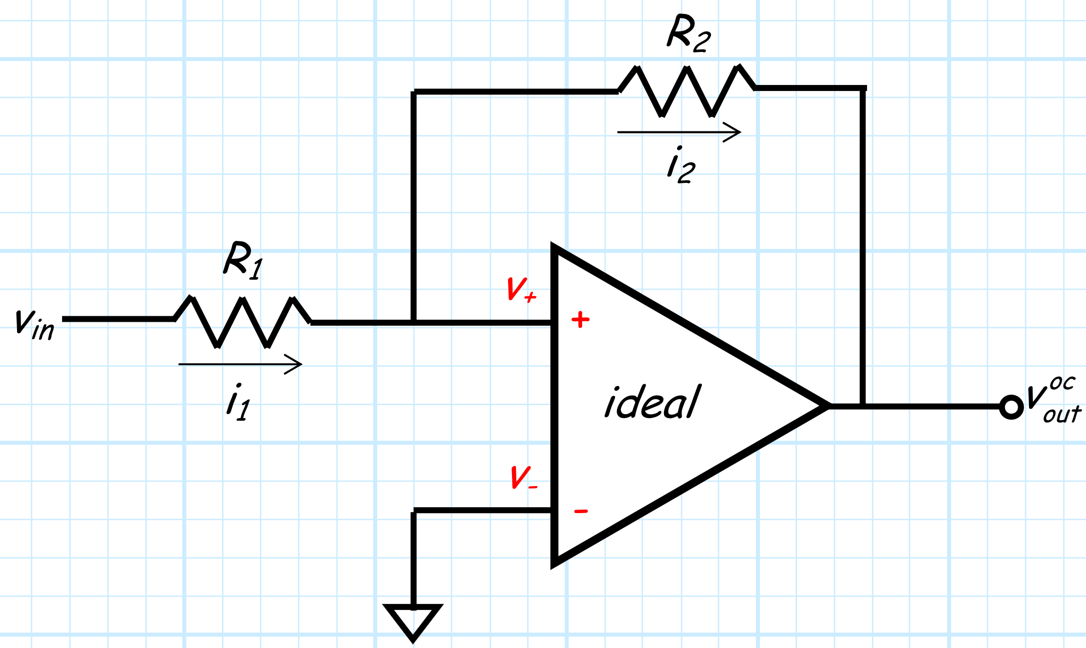
$$\frac{\partial v_{out}^{oc}}{\partial v_-}\frac{\partial v_-}{\partial v_{out}^{oc}} = -A_{op}\left(\frac{R_1}{R_1+R_2}\right)$$

is a **negative** value; we refer to this case as **negative feedback**.

Negative feedback keeps the inverting voltage **in place** (i.e., $v_-=0$)—it "enforces" the concept of the virtual ground!

Let's try some positive feedback

Contrast this behavior with that of the following circuit:



Q: *Isn't this precisely the **same** circuit as before?*

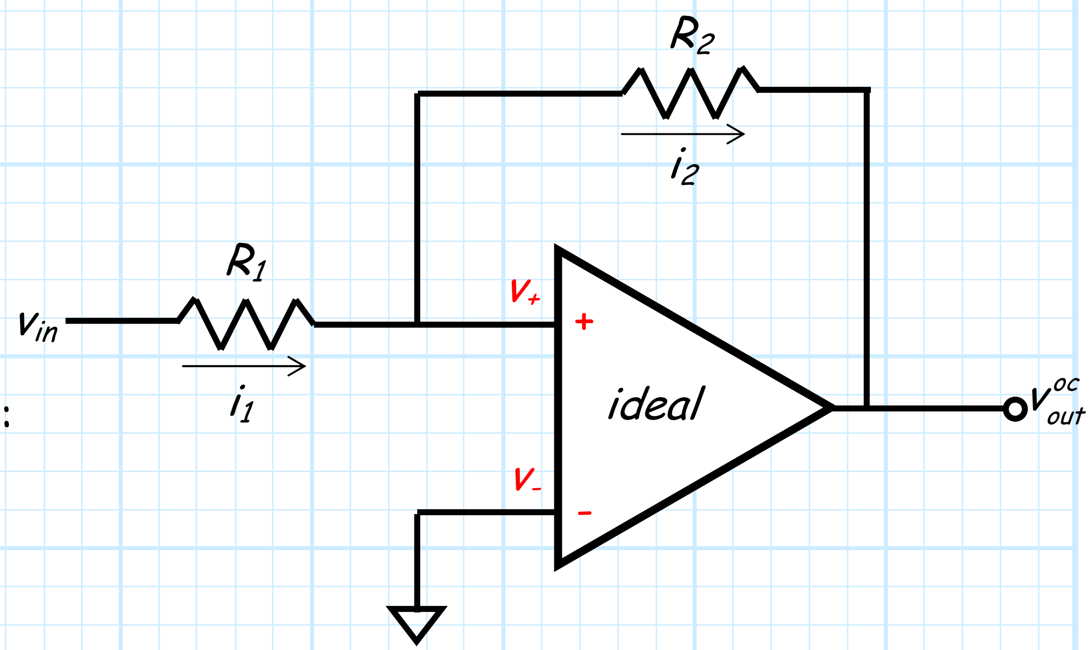
A: NO!

Note that the feedback resistor is now connected to the **non**-inverting terminal, and the **inverting** terminal is now grounded.

Positive derivatives!

The feed-**forward** equations for **this** circuit are thus:

$$v_{out}^{oc} = A_{op}\ v_+$$



And so:

$$\frac{\partial v_{out}^{oc}}{\partial v_+} = \frac{\partial \left(A_{op}\ v_+\right)}{\partial v_+} = A_{op}$$

while the feed-**back** equations are:

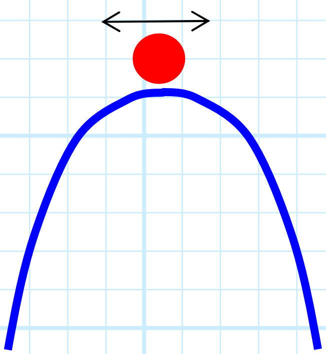
$$v_+ = \frac{R_2\, v_{in} + R_1\, v_{out}^{oc}}{R_1 + R_2}$$

and:

$$\frac{\partial v_+}{\partial v_{out}^{oc}} = \frac{\partial}{\partial v_{out}^{oc}}\left(\frac{R_2\, v_{in} + R_1\, v_{out}^{oc}}{R_1 + R_2}\right) = \frac{R_1}{R_1 + R_2}$$

Note in this case, **both** derivatives are **positive**.

Positive feedback-in this case it's a bad thing!



This means that an **increase** in v_+ will lead to an **increase** in v_{out}^{oc}. The problem is that the feedback will react by **increasing** v_+ even more—the error is **not** corrected, it is instead **reinforced**!

The result is that the output voltage will be sent to $v_{out}^{oc} = \infty$ or $v_{out}^{oc} = -\infty$ (i.e., the amplifier will **saturate**).

Note that the **loop product** for this case is **positive**:

$$\frac{\partial v_{out}^{oc}}{\partial v_+} \frac{\partial v_+}{\partial v_{out}^{oc}} = A_{op} \left(\frac{R_1}{R_1 + R_2} \right)$$

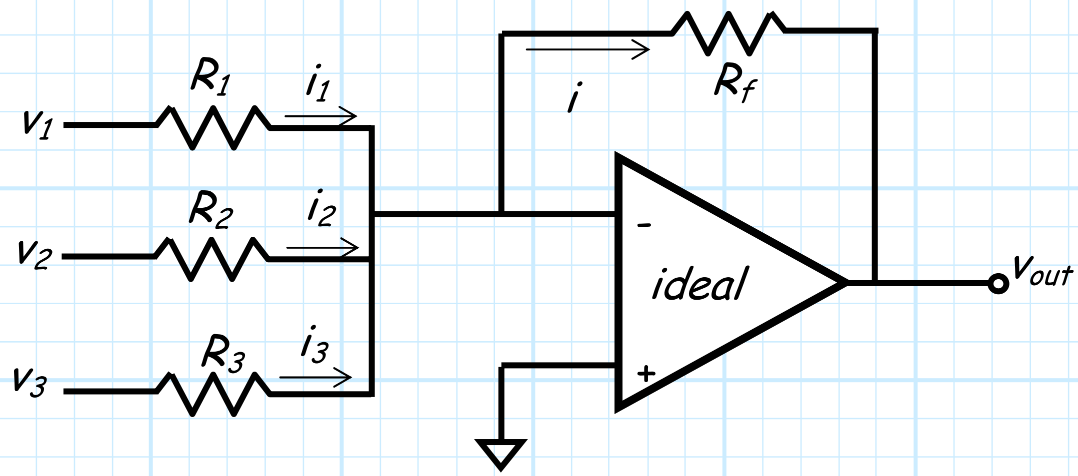
Thus, we refer to this case as **positive** feedback.

→ **Positive** feedback typically leads to amplifier **instability**!

As a result, we find that the **feed-back** portion of an op-amp circuit almost always is connected to its **inverting** (-) terminal!

The Weighted Summer

Consider an inverting amplifier with **multiple** inputs!



From **KCL**, we can conclude that the currents are related as:

$$i = i_1 + i_2 + i_3$$

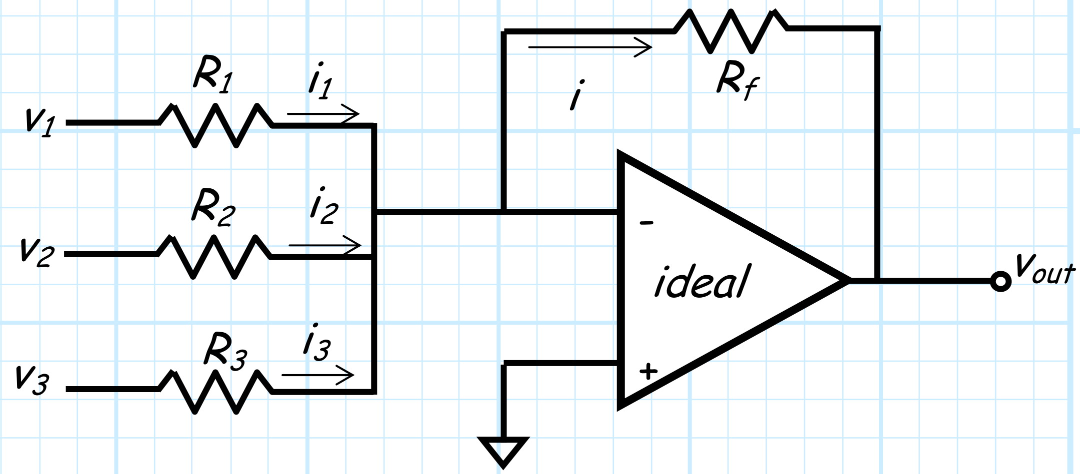
and because of **virtual ground** ($i_- = i_+ = 0$), we can conclude from **Ohm's Law**:

$$i_1 = \frac{v_1 - v_-}{R_1} = \frac{v_1}{R_1} \qquad i_2 = \frac{v_2 - v_-}{R_2} = \frac{v_2}{R_2} \qquad i_3 = \frac{v_3 - v_-}{R_3} = \frac{v_3}{R_3}$$

The output voltage

Likewise:

$$i = \frac{v_- - v_{out}}{R_f} = \frac{-v_{out}}{R_f}$$



Inserting these results into the initial KCL expression:

$$\frac{-v_{out}}{R_f} = \frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3}$$

Now, we sprinkle on some algebraic pixie dust, and find the output voltage:

$$v_{out} = -\frac{R_f}{R_1} v_1 - \frac{R_f}{R_2} v_2 - \frac{R_f}{R_3} v_3$$

The output is thus a weighted summation of each of the input signals!

We therefore refer to this circuit as the **weighted summer**.

How to combine signals

Note that if $R_f = R_1 = R_2 = R_3$, the output is an **unweighted summer**:

$$v_{out}(t) = -\left(v_1(t) + v_2(t) + v_3(t)\right)$$

For example, if:

$$v_1(t) = 2.0\cos(2\pi t + \pi)$$

$$v_2(t) = 1.0\cos(2\pi t + \pi/3)$$

$$v_3(t) = 1.5\cos(2\pi t - \pi/4)$$

then:

$$v_{out}(t) = -2.0\cos(2\pi t + \pi) - 1.0\cos(2\pi t + \pi/3) - 1.5\cos(2\pi t - \pi/4)$$

The summer is a method for **combining** several signals!