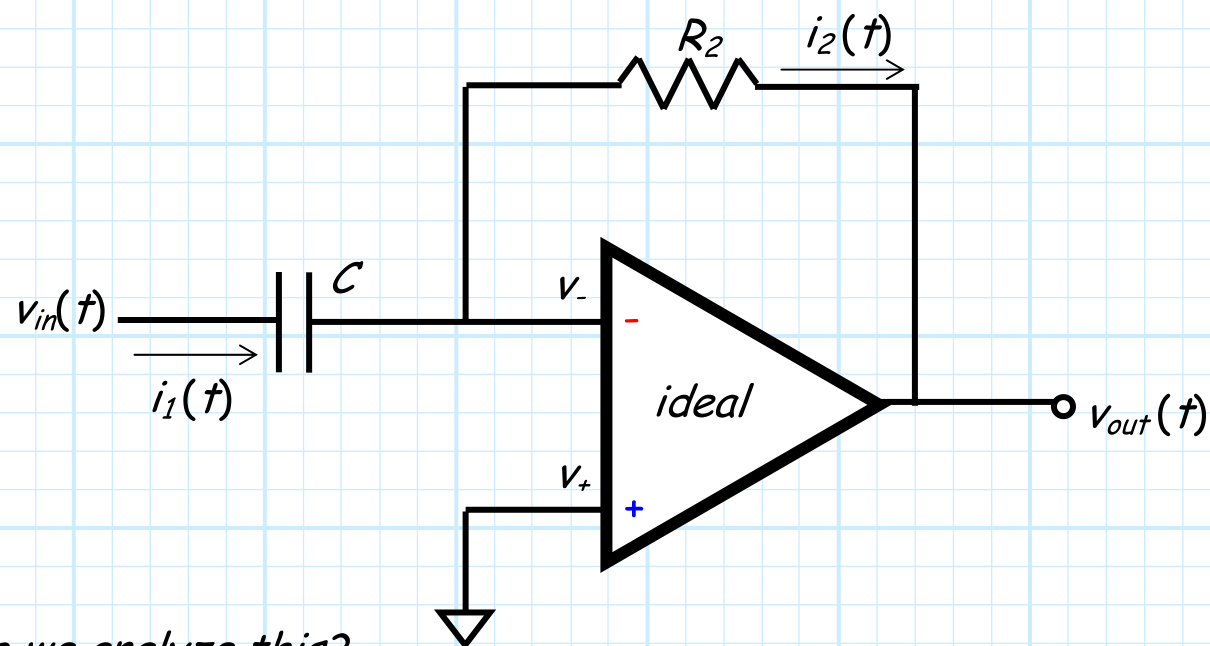
Op-Amp circuits with reactive elements

Now let's consider the case where the op-amp circuit includes **reactive elements**:



Q: *Yikes! **How** do we analyze this?*

A: Don't panic! Remember, the relationship between v_{out} and v_{in} is **linear**, so we can express the output as a **convolution**:

$$v_{out}(t) = \mathcal{L}\left[v_{in}(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_{in}(t')\, dt'$$

Just find the Eigen value

Q: *I'm **still** panicking—**how** do we determine the impulse response $g(t)$ of this circuit?*

A: Say the input voltage $v_{in}(t)$ is an **Eigen function** of linear, time-invariant systems:

$$v_{in}(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

Then, the **output** voltage is just a **scaled** version of this **input**:

$$v_{out}(t) = \mathcal{L}\left[e^{-st}\right] = \int_{-\infty}^{t} g(t-t')\, e^{-st} dt' = G(s)\, e^{-st}$$

where the "scaling factor" $G(s)$ is the complex **Eigen value** of the linear operator $\mathcal{L}$.

Express the input as a superposition of eigen values (i.e., the Laplace transform)

Q: *First of all, **how** could the input (and output) be this **complex** function e^{st}? Voltages are **real-valued**!*

A: True, but the **real-valued** input and output functions can be expressed as a weighted superposition of these **complex** Eigen functions!

$$V_{in}(s) = \int_0^{+\infty} v_{in}(t)\, e^{-st}\, dt$$

The **Laplace transform**→

$$V_{out}(s) = \int_0^{+\infty} v_{out}(t)\, e^{-st}\, dt$$

Such that:

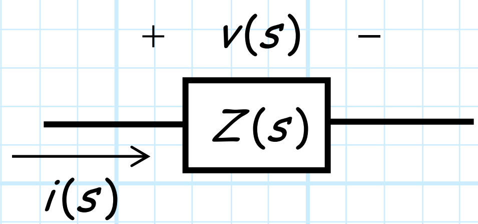
$$V_{out}(s) = G(s)\, V_{in}(s)$$

Find the eigen value from circuit theory and impedance

Q: *Still, I don't know **how** to find the eigen value $G(s)$!*

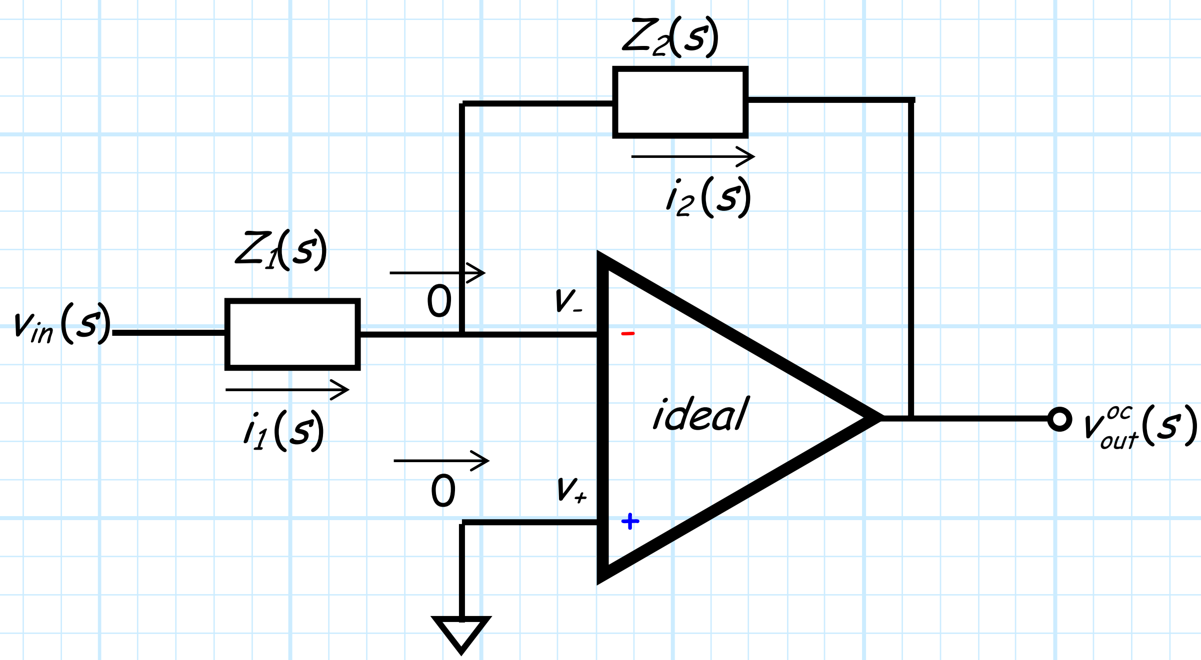
A: Remember, we can find $G(s)$ by analyzing the circuit using the Eigen value of **each linear circuit element**—a value we know as complex **impedance**!

$$\frac{v(s)}{i(s)} = Z(s)$$



For example

For **example**, consider this amplifier in with the **inverting configuration**, where the resistors have been **replaced** with complex **impedances**:



What is the **open-circuit voltage gain** $A_{vo}(s) = {v_{out}^{oc}(s)}/{v_{in}(s)}$?

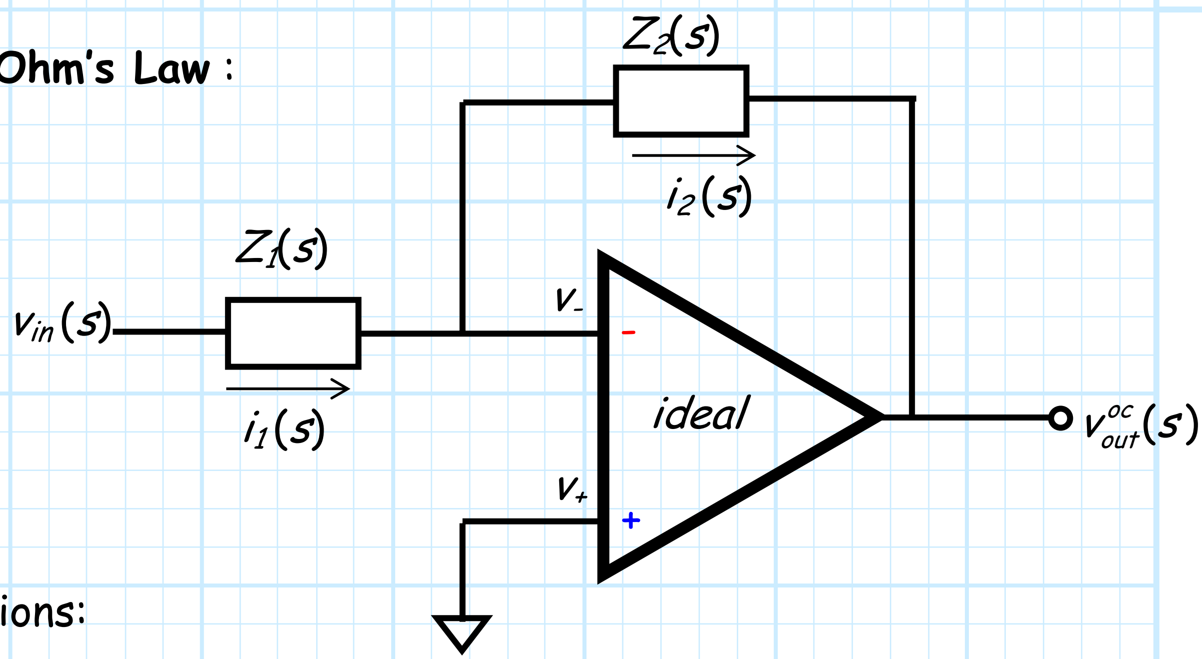
The eigen value of this linear operator

From **KCL**:

$$i_1(s) = i_2(s)$$

Since $v_-(s) = 0$, we find from **Ohm's Law** :

$$i_1(s) = \frac{v_{in}(s)}{Z_1(s)}$$



And also from **Ohm's Law**:

$$i_2(s) = \frac{-v_{out}^{oc}(s)}{Z_2(s)}$$

Equating the last two expressions:

$$\frac{v_{in}(s)}{Z_1(s)} = \frac{-v_{out}^{oc}(s)}{Z_2(s)}$$

Rearranging, we find the **open-circuit voltage gain**:

$$A_{vo}(s) = \frac{v_{out}^{oc}(s)}{v_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

The result passes the sanity check

Note that this complex voltage gain $A_{vo}(s)$ is the **Eigen value** $G(s)$ of the linear operator relating $v_{in}(t)$ and $v_{out}(t)$:

$$v_{out}(t) = \mathcal{L}\left[v_{in}(t)\right]$$

Note also that **if** the impedances $Z_1(s)$ and $Z_2(s)$ are real valued (i.e., they're resistors!):

$$Z_1(s) = R_1 + j0 \qquad \text{and} \qquad Z_2(s) = R_2 + j0$$

Then the voltage gain simplifies to the **familiar**:

$$A_{vo}(s) = \frac{V_{out}^{oc}(s)}{V_{in}(s)} = -\frac{R_2}{R_1}$$

Or, we can use the Fourier transform

Now, recall that the variable s is a **complex frequency**:

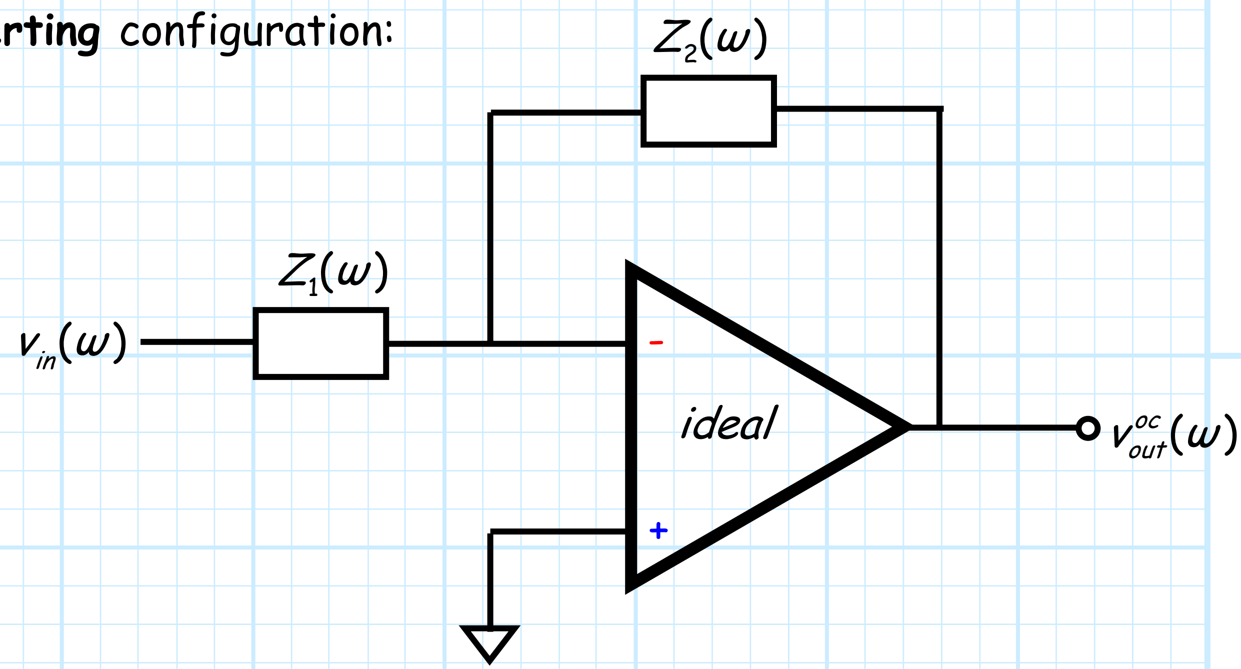
$$s = \sigma + j\omega .$$

If we set $\sigma = 0$, then $s = j\omega$, and the functions $Z(s)$ and $A_{vo}(s)$ in the Laplace domain can be written in the frequency (i.e., **Fourier**) domain!

$$A_{vo}(\omega) = A_{vo}(s)\big|_{\sigma=0}$$

And therefore, for the **inverting** configuration:

$$A_{vo}(\omega) = \frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)}$$



For the non-inverting

Likewise, for the non-inverting configuration, we find:

$$A_{vo}(\omega) = \frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)} = 1 + \frac{Z_2(\omega)}{Z_1(\omega)}$$

$$A_{vo}(s) = \frac{v_{out}^{oc}(s)}{v_{in}(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}$$