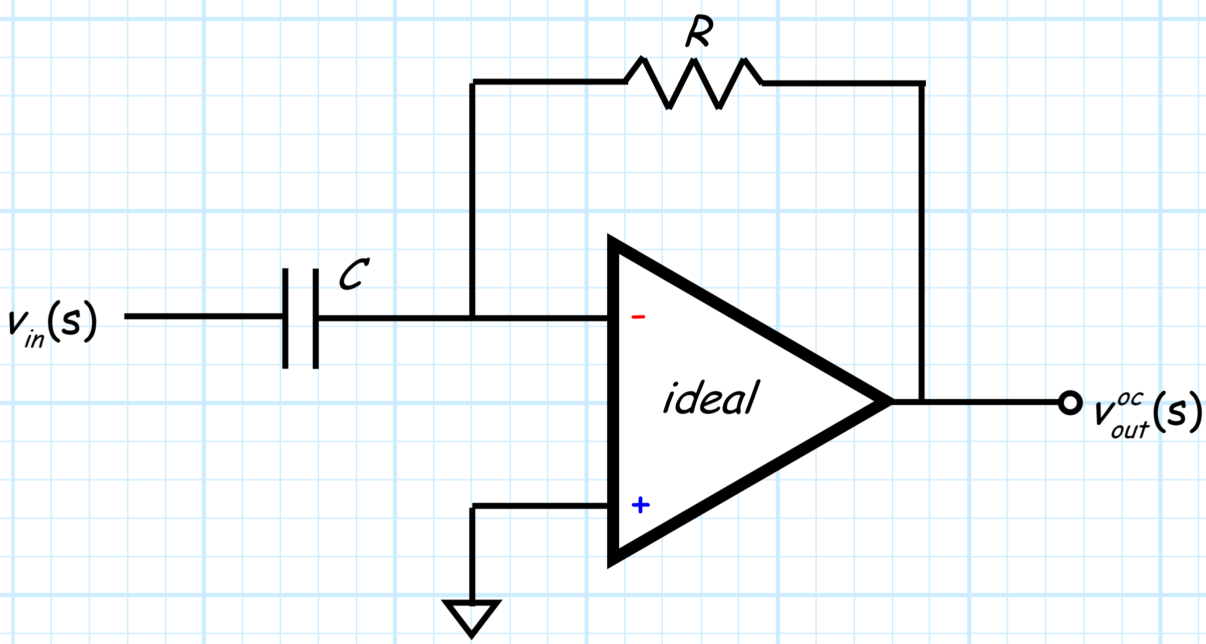
The Inverting Differentiator

The circuit shown below is the inverting **differentiator**.



Since the circuit uses the **inverting** configuration, we can conclude that the circuit transfer function is:

$$G(s) = \frac{v_{out}^{oc}(s)}{v_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Know the impedance; know the answer

For the **capacitor**, we know that its **complex impedance** is:

$$Z_1(s) = \frac{1}{sC}$$

And the complex impedance of the **resistor** is simply the real value:

$$Z_2(s) = R$$

Thus, the **eigen value** of the linear operator relating $v_{in}(t)$ to $v_{out}^{oc}(t)$ is:

$$G(s) = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R}{1/sC} = -s\,RC$$

In other words, the (Laplace transformed) **output** signal is related to the (Laplace transformed) **input** signal as:

$$v_{out}^{oc}(s) = -s\left(RC\right)\ v_{in}(s)$$

From our knowledge of **Laplace Transforms**, we know this means that the output signal is proportional to the **derivative** of the input signal!

Converting back to time domain

Taking the **inverse** Laplace Transform, we find:

$$v_{out}^{oc}(t) = -RC\,\frac{d\,v_{in}(t)}{d\,t}$$

For example, if the **input** is:

$$v_{in}(t) = \sin\omega t$$

then the **output** is:

$$\begin{aligned} v_{out}^{oc}(t) &= -RC\,\frac{d\,v_{in}(t)}{dt} \\ &= -RC\,\frac{d\,\sin\omega t}{dt} \\ &= -\omega\,RC\,\cos\omega t \end{aligned}$$

Or, with Fourier analysis

We likewise could have determined this result using **Fourier analysis** (i.e., frequency domain):

$$G(\omega)=\frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)}=-\frac{Z_2(\omega)}{Z_1(\omega)}=-\frac{R}{\left(1/j\omega C\right)}=-j\,\omega\,RC$$

Thus, the **magnitude** of the transfer function is:

$$\begin{aligned}\left|G(\omega)\right| &= \left|-j\omega\,RC\right|\\ &= \omega\,RC\end{aligned}$$

And since:

$$-j=e^{-j\left(\pi/2\right)}=\cos\left(-\pi/2\right)+j\sin\left(-\pi/2\right)$$

the **phase** of the transfer function is:

$$\begin{aligned}\angle G(\omega) &= -\pi/2 \quad \text{radians}\\ &= -90^\circ\end{aligned}$$

Look at the magnitude and phase

So given that:

$$\left|v_{out}^{oc}(\omega)\right| = \left|G(\omega)\right| \left|v_{in}(\omega)\right|$$

and:

$$\angle v_{out}^{oc}(\omega) = \angle G(\omega) + \angle v_{in}(\omega)$$

we find for the **input**:

$$v_{in}(t) = \sin \omega t$$

where:

$$\left|v_{in}(\omega)\right| = 1 \qquad \text{and} \qquad \angle v_{in}(\omega) = 0$$

that the **output** of the inverting differentiator is:

$$\left|v_{out}^{oc}(\omega)\right| = \left|G(\omega)\right| \left|v_{in}(\omega)\right| = \omega RC$$

and:

$$\angle v_{out}^{oc}(\omega) = \angle G(\omega) + \angle v_{in}(\omega) = -90^{\circ} + 0 = -90^{\circ}$$

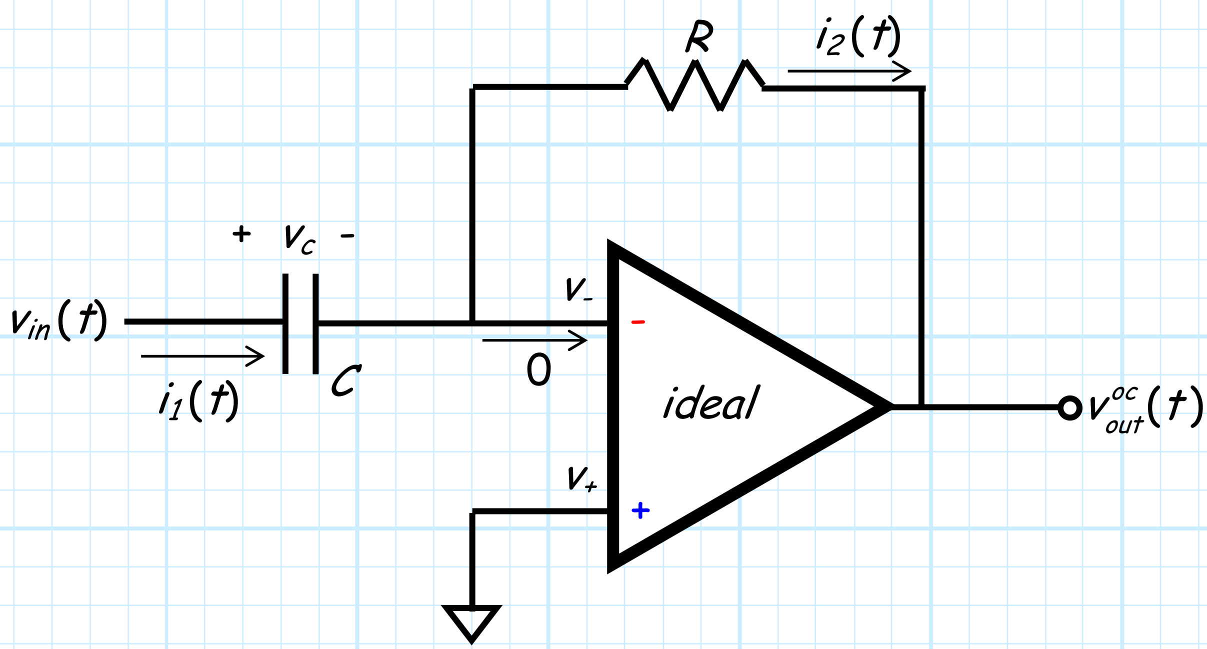
The result is the same!

Therefore, the output is:

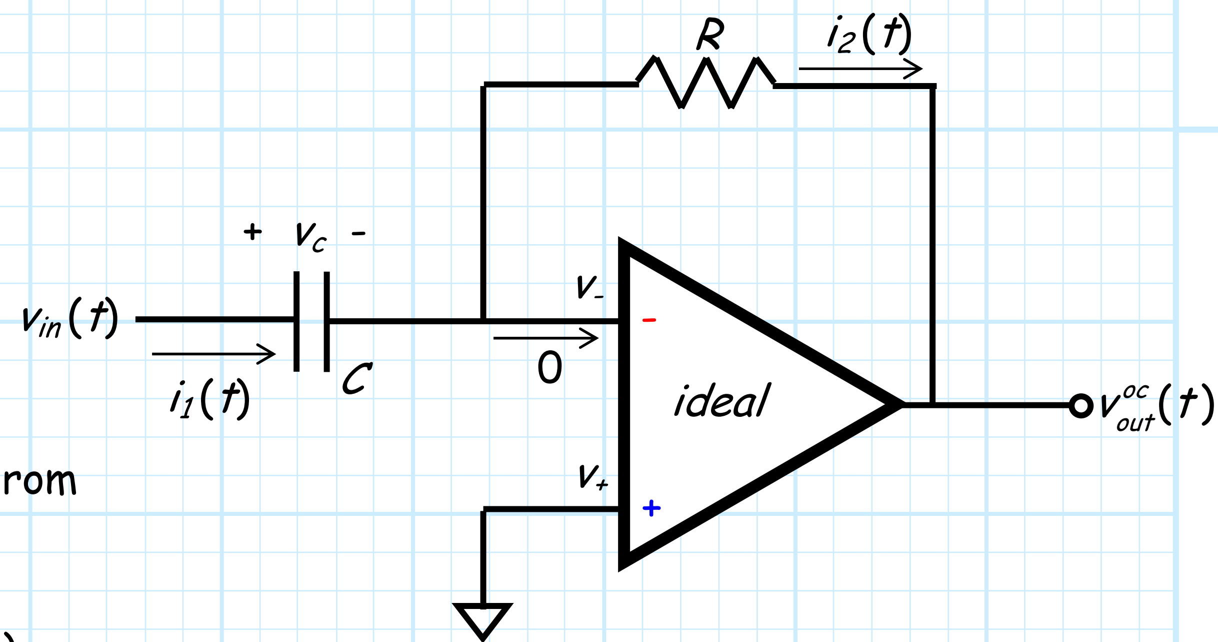
$$\begin{aligned} v_{out}^{oc}(t) &= \omega RC \sin\left(\omega t - 90^\circ\right) \\ &= -\omega RC \cos\omega t \end{aligned}$$

Exactly the **same result** as before (using Laplace trasforms)!

If you are still **unconvinced** that this circuit is a differentiator, consider this **time-domain** analysis.



Let's do a time-domain analysis



From our elementary **circuits knowledge**, we know that the current through a capacitor ($i_1(t)$) is:

$$i_1(t) = C \frac{d\, v_c(t)}{dt}$$

and from the circuit we see from KVL that:

$$v_c(t) = v_{in}(t) - v_-(t) = v_{in}(t)$$

therefore the **input current** is:

$$i_1(t) = C \frac{d\, v_{in}(t)}{dt}$$

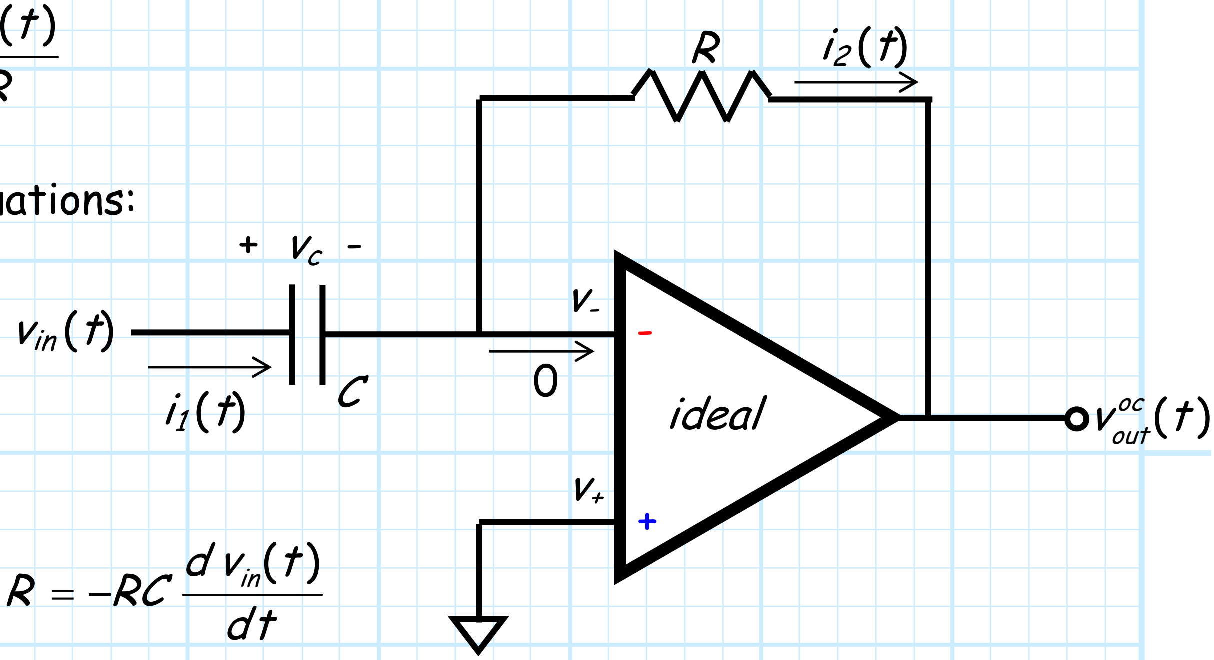
Laplace, Fourier, time-domain: the result it the same!

From KCL, we likewise know that:

$$i_1(t) = i_2(t)$$

and from Ohm's Law:

$$i_2(t) = \frac{v_1(t) - v_{out}^{oc}(t)}{R} = -\frac{v_{out}^{oc}(t)}{R}$$



Combining the two previous equations:

$$v_{out}^{oc}(t) = -i_1(t)\,R$$

and thus:

$$v_{out}^{oc}(t) = -i_1(t)\,R = -\left(C\,\frac{d\,v_{in}(t)}{dt}\right)R = -RC\,\frac{d\,v_{in}(t)}{dt}$$

The **same** result as before!